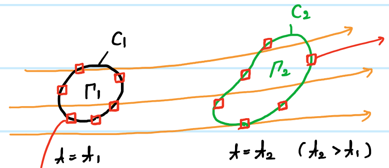
Kelvin's theorem

Wednesday, August 23, 2017 08:57

For the glory of God

Kelvin's Circulation Theorem

- Consider an arbitrary **inviscid** and incompressible flow as sketched below. (Assume all f are zero)



The same fluid elements at a later time t_2.

↳ The fluid elements now form a different curve C_2

Streamlines (move downstream)

Fluid elements along a curve C_1 at t_1

○ okay ⊙ no

↱ Here, the curve is simply connected region, i.e. single valued function

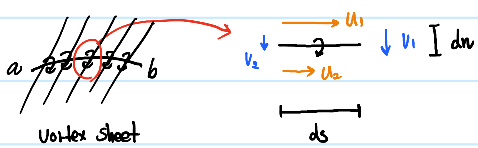
- Kelvin's theorem says that ; $\dfrac{D\Gamma}{Dt} = 0$; where $\Gamma = \oint_c \vec{v} \cdot d\vec{s}$ by definition

" The time rate of change of **circulation** around a closed curve consisting of same fluid element is zero "

$\Gamma = \iint_s (\nabla \times \vec{v}) \, d\vec{s}$; by Stokes theorem (Relation between circulation and vorticity)

Circulation

- It is a scalar integral quantity which is a measure of rotation for a finite area of fluid.



; Here, $\Gamma = \oint_c \vec{u} \cdot d\vec{s}$ ↻ ⊕ for the finite area

$= u_1 ds + v_1 dn - u_2 ds - v_2 dn$

$= (u_1 - u_2) ds + (v_1 - v_2) dn$

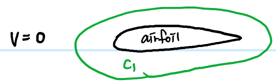
Tangential velocity discontinuity ← $= (u_1 - u_2) ds$ ↳ ≈ 0 if thin, $ds \ggg dn$ (or shrinking z axis)

$\therefore \quad \Gamma = \int_a^b \gamma(s) \, ds$ = strength of vortex sheet $= \gamma(s) \, ds$

↳ Strength of vortex sheet per unit length

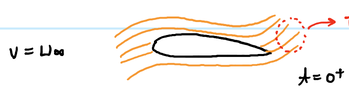
Kelvin's theorem around airfoil

- The theorem helps to explain the generation of circulation around an airfoil.
- To begin, consider an airfoil in a fluid at rest. ($\alpha = 2°$)



$\Gamma = \oint_{c_1} \vec{v} \cdot d\vec{s} = 0 \; (\because v = 0)$

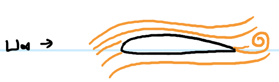
- Now, start the flow in motion over the airfoil.



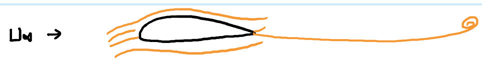
→ The flow tends to curl around the trailing edge. (Note that $\alpha = 2°$)

- As it moves downstream, it tends to roll up and form **a point vortex**

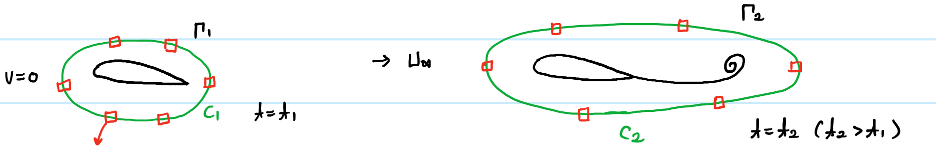
Called as starting vortex



- The starting vortex moves steadily downstream with the flow forever after.



· Now, let's talk about the fluid elements that initially made up as shown in below.



Here, the fluid elements have moved downstream and now make up curve 2 (C_2)

· From the Kelvin's theorem,

- The circulation Γ_2 around curve C_2 is the same as that around curve C_1, namely, $\Gamma_1 = \Gamma_2$

By the way, Does it make sense?

Since $\Gamma_1 = 0$, $\Gamma_1 = \Gamma_2 = 0$

· If the flow is assumed as an irrotational flow, we will be able to use Kutta-Joukowski theorem.

$$L' \text{ (sectional lift)} = \rho_\infty U_\infty \Gamma_c$$

→ Then, we have a zero lift for this case? Didn't you see that we have $\alpha = 2°$?

We will see what's going on there.

· Now, let us subdivide C_2 into two loops, thus forming C_3 and C_4.

C_3 : It encloses the starting vortex. (Counter clockwise)

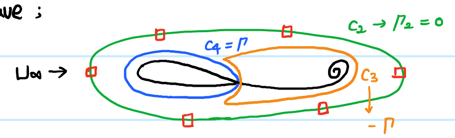
C_4 : It encloses the airfoil.

· If we consider the direction of each circulation, we have ;

$$\Gamma_2 = \Gamma_3 + \Gamma_4 \Leftrightarrow \Gamma_4 = -\Gamma_3$$

(when $\Gamma_2 = 0$)

$$\Leftrightarrow 0 = -\Gamma + \Gamma$$



The circulation around the airfoil is equal and opposite to the circulation around the starting vortex.

↳ This may be an answer for the question : How does nature generate the circulation?

cf. Kelvin's theorem also holds for an inviscid compressible flow in the special case where $\rho = \rho(p)$, that is, the density is some single-valued function of pressure.