

# Vorticity transport equation

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For the glory of God

- Vorticity transport equation is generally obtained by taking the curl of Navier-Stokes equation.

↳ It describes evolution of the  $\vec{\omega} = \nabla \times \vec{u}$  of a particle of a fluid as it moves with its flow.

- Navier-Stokes equation is as follows:

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{u} + \frac{\mu}{3} \nabla \Delta ; \text{ where } \Delta = \nabla \cdot \vec{u} \text{ and } \mu = \text{const.}$$

o (∴ Negligible body force)

- If we assume that the flow is incompressible,  $\Delta = 0$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u}$$

$$\Leftrightarrow \frac{D\vec{u}}{Dt} = -\nabla \left( \frac{p}{\rho} \right) + \nu \nabla^2 \vec{u} ; \text{ where } \frac{D}{Dt} = \frac{d}{dt} + \vec{u} \cdot \nabla$$

$$\Leftrightarrow \frac{d\vec{u}}{dt} + \vec{u} \cdot \nabla \vec{u} = -\nabla \left( \frac{p}{\rho} \right) + \nu \nabla^2 \vec{u}$$

By taking the curl of the equation, we have

$$\nabla \times \left( \frac{d\vec{u}}{dt} + \vec{u} \cdot \nabla \vec{u} \right) = \nabla \times \left[ -\nabla \left( \frac{p}{\rho} \right) + \nu \nabla^2 \vec{u} \right]$$

Vector Identity

$$\Leftrightarrow \nabla \times \frac{d\vec{u}}{dt} + \nabla \times (\vec{u} \cdot \nabla \vec{u}) = -\nabla \times \nabla \left( \frac{p}{\rho} \right) + \nabla \times (\nu \nabla^2 \vec{u}) ; \text{ where } \nabla \times \nabla^2 \vec{u} = \nabla^2 (\nabla \times \vec{u})$$

o if  $\rho = \text{const}$  (∴  $\nabla \times \nabla p \equiv 0$ ) : Pressure term = 0

$$\Leftrightarrow \frac{D\vec{u}}{Dt} + \vec{u} \cdot \nabla \vec{u} + (\nabla \times \vec{u}) \cdot \nabla \vec{u} = \nu \nabla^2 \vec{u}$$

z direction      in x-y plane for  
↓                  ↓                  2D flow

$$\Leftrightarrow \frac{D\vec{u}}{Dt} + \vec{u} \cdot \nabla \vec{u} = \nu \nabla^2 \vec{u}$$

Here  $\vec{a} = \vec{u}$ ,  $\vec{b} = \nabla \vec{u}$  (i.e.  $\nabla \vec{u}$ )

o if two-dimensional flow ( $\vec{a} \cdot \vec{b} = 0$  if  $\vec{a} \perp \vec{b}$ )

Vortex stretching

(Important in turbulent flow) To be more specific,  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos\theta$

Hence, we have:  $= 0 \text{ if } \theta = 90^\circ$

$$\frac{D\vec{u}}{Dt} = \nu \nabla^2 \vec{u} \quad \rightarrow \text{Viscous diffusion of vorticity}$$

(= 0 for inviscid flow) : Viscous term  $\neq 0$  for viscous flow

- What if we have compressible flow?

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{u} + \frac{\mu}{3} \nabla \Delta ; \text{ where } \Delta = \nabla \cdot \vec{u} \text{ and } \mu = \text{const.}$$

o (∴ Negligible body force)

$$\rho \frac{D\vec{U}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{U} + \frac{\mu}{3} \nabla \Delta ; \text{ where } \Delta = \nabla \cdot \vec{U} \text{ and } \mu = \text{const.}$$

0 (∴ Negligible body force)

$$\Leftrightarrow \frac{D\vec{U}}{Dt} + \vec{U} \cdot \nabla \vec{U} = - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{U} + \frac{\nu}{3} \nabla \Delta$$

By taking curl of the equation, we have;

$$\nabla \times \left( \frac{D\vec{U}}{Dt} + \vec{U} \cdot \nabla \vec{U} \right) = \nabla \times \left( - \frac{1}{\rho} \nabla p \right) + \nabla \times (\nu \nabla^2 \vec{U}) + \nabla \times \left( \frac{\nu}{3} \nabla \Delta \right)$$

From the vector identity,  $\vec{U} \cdot \nabla \vec{U} \equiv \frac{1}{2} \nabla (\vec{U} \cdot \vec{U}) - \vec{U} \times \vec{\omega}$

$$\Leftrightarrow \frac{D\vec{\omega}}{Dt} + \frac{1}{2} \nabla \times \nabla (\vec{U} \cdot \vec{U}) - \nabla \times (\vec{U} \times \vec{\omega}) = - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nu \nabla^2 \vec{\omega} + \frac{\nu}{3} \nabla \times (\nabla \cdot \vec{U})$$

0 (∴  $\nabla \times \nabla \phi = 0$ )

$$\Leftrightarrow \frac{D\vec{\omega}}{Dt} = \nabla \times (\vec{U} \times \vec{\omega}) - \nabla \times \left( \frac{\nabla p}{\rho} \right) + \nu \nabla^2 \vec{\omega}$$

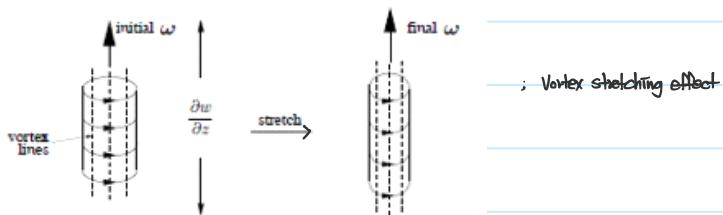
$$= \frac{1}{\rho} \nabla \times (\nabla p) - \nabla \left( \frac{1}{\rho} \right) \times \nabla p$$

0 (∴  $\nabla \times \nabla \phi = 0$ )

$$= - \left( \frac{\rho \nabla \phi - \nabla \rho}{\rho^2} \times \nabla p \right)$$

$$= - \frac{\nabla \rho \times \nabla p}{\rho^2}$$

$$\Leftrightarrow \frac{D\vec{\omega}}{Dt} = \nabla \times (\vec{U} \times \vec{\omega}) - \frac{\nabla \rho \times \nabla p}{\rho^2} + \nu \nabla^2 \vec{\omega} ; \text{ Thus, we have pressure term.}$$



2017 Fall question from P.K. Young

Let's write down vorticity transport equation (Incompressible)

Answer)

$$\rho \frac{D\vec{U}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{U} + \frac{\mu}{3} \nabla \Delta$$

$$\Leftrightarrow \frac{D\vec{U}}{Dt} = - \nabla \left( \frac{p}{\rho} \right) + \nu \nabla^2 \vec{U}$$

By taking curl,

$$\nabla \times \frac{D\vec{w}}{Dt} = - \nabla \times \nabla \left( \frac{P}{\rho} \right) + \nabla \times \nu \nabla^2 \vec{w}$$

$\circ$  ( $\because \nabla \times \nabla P \equiv 0$ )

$$\Leftrightarrow \frac{D\vec{w}}{Dt} = \nu \nabla^2 \vec{w}$$

Based on it, pressure term was disappeared. But it's a fact that vorticity is always affected by pressure.

How would you explain this?

- Yes, it is true; however, if you look at LHS

$$\frac{D\vec{w}}{Dt} = \frac{d\vec{w}}{dt} + \vec{\omega} \cdot \nabla \vec{w}$$

$\hookrightarrow$  There is a term in terms of Velocity.

- Hence, the pressure effect could be explained by the velocity term, i.e. convective and  $P \& U$  relationship.