

Compressible boundary Layer

Thursday, September 21, 2017 20:39

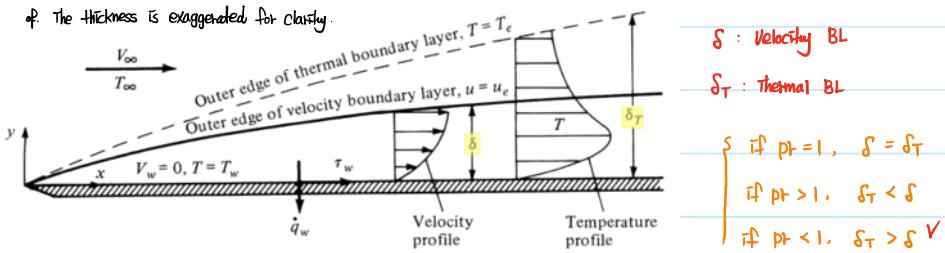
For the glory of God

Introduction

- The concept of the Boundary layer (BL) was first introduced by Ludwig Prandtl in 1904.
- It allowed the practical calculation of drag and flow separation over Aerodynamic bodies.
- Then, what is **boundary layer** again?

↳ It is the thin region of flow adjacent to a surface where the flow is retarded by the influence of friction between a solid surface and the fluid.

↳ The thickness is exaggerated for clarity.



✓ $\Pr = 0.71$ (standard condition)

δ : Velocity BL

δ_T : Thermal BL

|
if $\Pr = 1$, $\delta = \delta_T$
if $\Pr > 1$, $\delta_T < \delta$
if $\Pr < 1$, $\delta_T > \delta$ ✓

- Although the boundary layer occupies geometrically only a small portion of the flow field, its influence on the drag and heat transfer to the body is immense - it produces marked results.

Governing Equations

- Recall that Boundary layer equations for incompressible and two-dimensional laminar flow with $\alpha = \text{const}$ / negligible body force,

we have :

$$\text{Continuity} : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$x\text{-momentum} : u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$y\text{-momentum} : - \frac{1}{\rho} \frac{dp}{dy} = 0 \quad \begin{matrix} \hookrightarrow \text{In BL approximation, } p = p(x) \text{ only} \\ \downarrow \frac{dp}{dy} = 0 \quad (\because \rho = \text{const}) ; \text{ It can be also shown experimentally.} \end{matrix}$$

Note that the energy equation is not needed at this point. However :

- In Compressible Boundary Layer, an acceleration is important and continuity is not trivial.
- Density becomes an important variable in Compressible flow
- Since density is not constant in this flow, Energy equation is introduced.

e.g. For incompressible flow, 4 variables (P, u, v, w) and 4 equations (Continuity and Momentum)

For compressible flow, 6 variables (P, u, v, w, ρ, T) and 6 equations

(Eq. of state)

Continuity, Momentum, Energy, and $P = \rho RT$

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- Note that, for a compressible boundary layer.

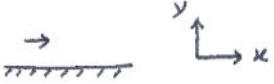
In compressible boundary layer.

- The Energy Equation must be included.
- We consider local property variation : ρ, μ
- The density is treated as a variable.
- In general, μ and K are functions of temperature and hence must be treated as variables.
- Hence, the energy equation under the assumption is as following :

§. Thermal Boundary Layers

a) Introduction

- We have discussed about not only conservation of Energy but also temperature stuffs.
- You may be wondering



- What is function of Temperature $T(y)$?
- What is $\frac{dT}{dy}$ at wall? in B.L.
- We will end up looking at the fact that temperature at wall is constant either mathematically or physically.

$\left. \begin{array}{l} T_b = \text{const} \\ T_w = \text{const} \end{array} \right\}$

b) Enthalpy Equation Simplification

- For a steady, two-dimensional, and laminar thermal boundary layer, the enthalpy equation can be simplified;

$$\rho C_p \left(u \frac{dT}{dx} + v \frac{dT}{dy} \right) = h \frac{dp}{dx} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{du}{dy} \right)^2$$

This is the boundary layer Energy equation.

Let's see how it worked out.

It was derived from the Enthalpy equation;

Let's see how it worked out.

It was derived from the Enthalpy equation:

(In order to derive the Enthalpy equation, we need to see K.E. and internal energy first.)

c) Kinetic energy Equation of Homework 7, 1 (a)

As we know, Kinetic energy equation will look like : Rate of K.E.
per unit mass

$$\frac{D}{Dt} \left(\frac{1}{2} U_i U_i \right) \text{ in tensor form}$$

By expanding out the material derivative, we have

$$\frac{D}{Dt} \left(\frac{1}{2} U_i U_i \right) = U_i \frac{DU_i}{Dt}$$

Now, let's dive into the derivation of the equation.

$$\frac{D}{Dt} \left(\frac{1}{2} U_i U_i \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} U_i U_i \right) + U_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} U_i U_i \right)$$

$$* = U_i \frac{\partial U_i}{\partial t} + U_j U_i \frac{\partial U_i}{\partial x_j}$$

$$= U_i \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right)$$

* = $U_i \frac{DU_i}{Dt}$; This says the kinetic energy equation can be obtained by simply multiplying the momentum \underline{u} by U_i

$$* y = [f(x)]^n \rightarrow y' = n f(x)^{n-1} \cdot f'(x) \quad \text{in the form of}$$

$$\text{Likewise, } \frac{1}{2} \frac{\partial}{\partial t} U_i^2 = \frac{1}{2} \cdot 2 U_i^{2-1} \frac{\partial U_i}{\partial t} = U_i \frac{\partial U_i}{\partial t}$$

$$\frac{\partial U_i}{\partial t} = \dots$$

$$* \frac{D(\cdot)}{Dt} = \frac{d(\cdot)}{dt} + \underline{u} \cdot \nabla \underline{u} \quad (\cdot) = \frac{1}{2}$$

d) Internal energy Equation of Homework 7, 1(b)

- In order to obtain internal energy equation, the math idea of derivation will look like;

$$\text{Internal energy} = \text{Total energy} - \text{kinetic energy}$$

$$\rho \frac{D\epsilon}{Dt} = \rho \frac{D}{Dt} \left(\epsilon + \frac{1}{2} \vec{u} \cdot \vec{u} \right) - \rho \frac{D}{Dt} \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right) ; \text{Take a look the derivation first}$$

- As a result,

$$\rho \frac{D\epsilon}{Dt} = Z_{JT} \frac{\partial U_T}{\partial X_J} + \frac{\partial}{\partial X_T} \left(K \frac{\partial T}{\partial X_T} \right)$$

- Now let's see how it worked out
- The derivation process would be done if we consider
 - Recall momentum equation in tensor form
 - Multiply by U_T \hookrightarrow to make K.E. term
 - Subtract the result equation off from total Energy eq.

Let's recall the momentum Equation.

$$\rho \frac{DU_i}{Dt} = \rho f_i + \frac{\partial Z_{JT}}{\partial X_J} \quad \text{Here, } Z_{JT} \text{ term didn't expand}$$

• By multiplying U_T , we have

$$U_T \rho \frac{DU_T}{DT} = U_T \rho f_T + U_T \frac{\rho Z_{JT}}{JF}$$

• If we take the result from Kinetic Energy Equation,

$$\rho \frac{D}{DT} \left(\frac{1}{2} U_T U_i \right) = U_T \rho f_T + U_i \frac{\rho Z_{JT}}{JF} \quad (\because \rho U_T \frac{DU_T}{DT} = \rho \frac{D}{DT} \left(\frac{1}{2} U_T U_i \right))$$

$$* = \rho U_i f_T + \frac{\rho}{JF} (U_i Z_{JT}) - Z_{JT} \frac{\rho U_i}{JF}$$

$$f \frac{dg}{dx} = \frac{d}{dx} (fg) - g \frac{df}{dx}$$

• Finally,

$$e = e_{tot} - K.E.$$

It is from the Energy equation.

$$\rho \frac{De}{DT} = \left\{ \rho f_T U_T + \frac{\rho}{JF} (Z_{JT} U_T) + \frac{\rho}{JF} (K \frac{dT}{dX_T}) \right\}$$

$$- \left\{ \rho f_T U_i + \frac{\rho}{JF} (Z_{JT} U_i) - Z_{JT} \frac{\rho U_i}{JF} \right\}$$

Hence,

$$\rho \frac{De}{DT} = \frac{\rho}{JF} \left(K \frac{dT}{dX_T} \right) + \underline{Z_{JT} \frac{\rho U_i}{JF}}$$

Didn't expand at all

Just dummy subscripts

- It's time to expand the term for Newtonian fluid.

$$Z_{JT} - \frac{dU_T}{dx_J} = Z_{JT} (S_{IJ} + t_{IJ})$$

$$= Z_{IJ} S_{IJ} + Z_{IJ} t_{IJ}$$

$$\begin{aligned} & \therefore Z_{IJ} = -Z_{JI} \\ & \therefore Z_{IJ} t_{IJ} = -Z_{JI} t_{IJ} \end{aligned}$$

$$\therefore Z_{IJ} t_{IJ} = 0$$

t_{IJ} is only solution to satisfy it.

$$\therefore \text{where } Z_{IJ} = Z_{JI}$$

For more detail..

Please note that we had already discussed about:

$$\frac{dU_T}{dx_J} = S_{IJ} (\text{Strain rate}) + R_{IJ} (\text{Rotation rate})$$

$$Z_{IJ} = Z_{JI} \quad (\text{Symmetric})$$

$$t_{IJ} = -t_{JI} \quad (\text{Anti-symmetric})$$

(\therefore Symmetric & Anti-)

$$Z_{IJ} t_{IJ} \equiv Z_{IJ} t_{JI} = Z_{IJ} (-t_{IJ}) = -Z_{IJ} t_{IJ}$$

Rename dummy subscripts

$$\text{Hence, } Z_{IJ} t_{IJ} = -Z_{JI} t_{IJ} = 0$$

Also, we know that (for Newtonian fluid):

$$Z_{JT} = Z_{IJ} = -P \delta_{IJ} + 2\mu (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ})$$

Back to the equation,

$$Z_{JT} \frac{dU_T}{dx_J} = Z_{IJ} S_{IJ}$$

\hookrightarrow It doesn't guarantee:

Hence, of. Homework 7, 1(c)

(cf. no division operation is defined for vector and matrix) $\frac{dU_T}{dx_J} = S_{IJ}$

\therefore we can't divide by Z_{IJ} \rightarrow This is because $Z_{IJ} t_{IJ} = 0$ doesn't mean $t_{IJ} = 0$

$$Z_{JT} \frac{dU_T}{dx_J} = Z_{IJ} S_{IJ} = S_{IJ} \left\{ -P \delta_{IJ} + 2\mu (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ}) \right\}$$

$$= -P S_{IJ} \delta_{IJ} + 2\mu S_{IJ} (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ})$$

$$= -P \Delta + 2\mu S_{IJ} (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ})$$

$$= -P \Delta + \mu \bar{\epsilon} \quad ; \text{ where } \bar{\epsilon} \geq 0$$

where $\Delta = \text{Strain rate}, \bar{\epsilon} = 2 S_{IJ} (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ})$

(≥ 0)

$2\mu S_{IJ} (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ}) = 2\mu S_{IJ} S_{IJ} - \frac{2}{3} \mu S_{IJ} \Delta \delta_{IJ}$, we don't know they end up \oplus or \ominus yet (in the end)

• Using the definition (Hw2), $d_{IJ} = S_{IJ} - \frac{1}{3} \Delta \delta_{IJ} \Leftrightarrow S_{IJ} = d_{IJ} + \frac{1}{3} \Delta \delta_{IJ}$

Finally, we will see

• Then, $2\mu (d_{IJ} + \frac{1}{3} \Delta \delta_{IJ}) d_{IJ} = 2\mu d_{IJ} d_{IJ} + \frac{2}{3} \mu \Delta \delta_{IJ} d_{IJ} \Rightarrow d_{IJ} \Leftrightarrow d_{IJ} = S_{IJ} - \frac{1}{3} \Delta \delta_{IJ} = \Delta - \frac{1}{3} \Delta \delta_{IJ} = 0$ \therefore It is ≥ 0

\therefore (≥ 0) monotonic in sign

\therefore Viscous shear tends to cause increase in ϵ (Dissipation of KE into heat)

• As we substitute the result into the Internal Energy Equation,

$$\rho \frac{De}{dt} = -P \Delta + \mu \bar{\epsilon} + \frac{d}{dx_I} (K \frac{dT}{dx_I})$$

- . Using the definition (Hw2), $d\sigma = S_{IJ} - \frac{1}{3} \Delta S_{IJ} \Leftrightarrow S_{IJ} = d\sigma + \frac{1}{3} \Delta S_{IJ}$
 Then, $2\mu (d\sigma + \frac{1}{3} \Delta S_{IJ})d\sigma = 2\mu d\sigma d\sigma + \frac{2}{3} \mu \Delta S_{IJ} d\sigma \rightarrow d\sigma \Leftrightarrow d\sigma = S_{IJ} - \frac{1}{3} \Delta S_{IJ} = \Delta - \frac{1}{3} \Delta^2 = 0$ Finally, we will see
that $\Delta \geq 0$
 $\therefore (\Delta) \text{ monotonic in sign}$ $\therefore \text{viscous stress tends to cause increase in } \Delta \text{ (Dissipation of KE into heat)}$
- . As we substitute the result into the Internal Energy Equation,

$$\rho \frac{de}{dt} = -p\Delta + \mu \Phi + \frac{d}{dx_I} \left(K \frac{dT}{dx_I} \right)$$

Here are some observations ;

1) $\Delta < 0$ (If being compressed), $-p\Delta > 0$ (work done on fluid)

2) If we think about viscous effect, we expect to see $\Phi \gg 0$

e) Enthalpy Equation of Homework, 1 (d)

In general, the Enthalpy is defined as following ;

$$\text{Enthalpy (H)} = \text{Internal energy (e)} + \text{flow work (PV)}$$

If we consider the enthalpy per unit mass, $v \text{ (specific volume)} = \frac{1}{\rho}$

$$h = e + \frac{P}{\rho} \quad ; \text{ where } v = \frac{m}{\rho} \rightarrow dv = \frac{1}{\rho} \frac{dm}{\rho}$$

$$\text{Then, } dh = de + d\left(\frac{P}{\rho}\right)$$

$$= de + \frac{\rho dp - Pdv}{\rho^2}$$

In order to derive the Enthalpy Equation, let's rearrange the equation and use the concept of substantial derivative.

• By multiplying ρ ,

$$\rho \frac{dh}{dt} = \rho \frac{De}{dt} + \frac{DP}{dt} - \frac{\rho}{\rho} \frac{DP}{dt}$$

• If we substitute $\rho \frac{De}{dt}$ equation into the equation above,

$$\begin{aligned}\rho \frac{dh}{dt} &= -\rho \Delta + \mu \bar{e} + \frac{d}{dx_T} (k \frac{\partial T}{\partial x_T}) + \frac{DP}{dt} - \frac{\rho}{\rho} \frac{DP}{dt} \\ &= \mu \bar{e} + \nabla \cdot (k \nabla T) + \frac{DP}{dt} - \rho \Delta - \frac{\rho}{\rho} \frac{DP}{dt} \\ &= \mu \bar{e} + \nabla \cdot (k \nabla T) + \frac{DP}{dt} - \rho \left(\Delta + \frac{1}{\rho} \frac{DP}{dt} \right)\end{aligned}$$

* $\rightarrow 0$ (Continuity Equation)

$$\therefore \rho \frac{dh}{dt} = \frac{DP}{dt} + \nabla \cdot (k \nabla T) + \mu \bar{e}$$

* From continuity equation, we have

$$\frac{DP}{dt} + \rho \nabla \cdot \vec{u} = 0 \Leftrightarrow \frac{DP}{dt} = -\rho \Delta$$

• If calorically perfect gas, $h = c_p T$; where $c_p = \text{constant}$

$$\rho c_p \frac{dT}{dt} = \frac{DP}{dt} + \nabla \cdot (k \nabla T) + \mu \bar{e}$$

Okay, let's derive the boundary layer energy equation (under the assumptions)

(we're going to apply this to boundary layer)

Starting from the Enthalpy Equation:

$$\nabla \cdot K \nabla T \Rightarrow K \nabla^2 T$$

$$\rho C_p \frac{dT}{dt} = \frac{DP}{dt} + K \nabla^2 T + \mu \overline{\dot{E}} ; \text{ where assume } K = \text{const}$$

convection compressibility heat conduction Viscous dissipation

If we expand all terms, we might have

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \left(\frac{\partial P}{\partial t} + u \frac{\partial P}{\partial x} + v \frac{\partial P}{\partial y} \right)$$

$$+ K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + 2\mu S_{TJ} (S_{TJ} - \frac{1}{3} \Delta \delta_{TJ})$$

* Assumptions

; where in terms of $\mu \overline{\dot{E}}$ term, : steady, 2D, const K and μ

$$2\mu S_{TJ} (S_{TJ} - \frac{1}{3} \Delta \delta_{TJ}) = 2 (S_{TJ}^2 - \frac{1}{3} \Delta \delta_{TJ} S_{TJ}) \mu$$

$$* = 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 - \frac{1}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right]$$

" S_{11}^2 " " S_{22}^2 " " S_{12}^2 and S_{21}^2 " " Δ^2 "

$$* S_{TJ} = \frac{1}{2} \left(\frac{\partial u_T}{\partial x_T} + \frac{\partial v_T}{\partial x_T} \right) = S_{JT}$$

If we consider the assumptions for Laminar boundary layer,

This is due to

the assumptions at the beginning. \rightarrow - steady $\frac{\partial}{\partial t} = 0$ - Flat plate flow $u \neq 0, v = 0$ at wall, $\frac{\partial v}{\partial y} = 0$:

This is due to

- negligible $\frac{\partial P}{\partial y} \approx 0$

assumptions for BL.
(of course, it's laminar)

- Temperature remains mostly in y $| \frac{\partial^2 T}{\partial y^2} | \gg | \frac{\partial^2 T}{\partial x^2} | \approx 0$

$$\frac{\partial u}{\partial y} \approx 0, \frac{\partial v}{\partial x} = 0$$

Hence, we have the Boundary layer Enthalpy equation. (under the assumptions)

$$\rho C_p (u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}) = u \frac{\partial P}{\partial x} + K \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 ; \text{ where } P = P_0 \text{ because of } y\text{-momentum equation}$$

Transformation (and self-similar solution)

- The boundary layer equation can be solved by using the concept of similarity; however, the solution would depend on whether the flow is laminar or turbulent.

↳ Keep in mind that self-similar solutions occur only for certain special types of flows, i.e.

the flow over a flat plate is one such example. In general, for the flow over an arbitrary body,

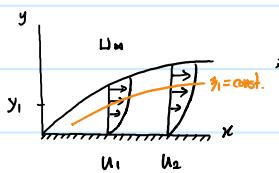
the B.L. solutions are non-similar; the P.D.E. cannot be reduced to O.D.E.

- In order to transform to O.D.E., the similarity parameter (η) must be introduced.

→ η is used to reduce the number of variables so that P.D.E. could be O.D.E.

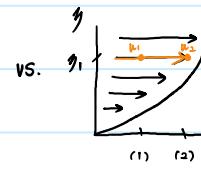
→ ... ; Momentum diffusion due to viscous effects is expected more as time

is increasing. (viscous length scale, \sqrt{vt}) The idea: Velocity profile at different location or times are similar.



$$\text{; where } u = f(x, y)$$

$$u_1(x_1, y_1) \neq u_2(x_2, y_2)$$



$$\text{; where } \frac{u}{U_{\infty}} = f(\eta)$$

$$u_1(x_1) = u_2(x_2)$$

Difficult y but same ratio

* I am : Is it close to the wall?

Thus, η can be defined as: Stokes suggested it $\rightarrow \eta$ (dimensionless) $= \frac{y}{\text{Something}}$: That would be better measure \downarrow it might be vague.

$$\eta = \frac{\text{distance from plate}}{\text{viscous length scale}} = \frac{y}{2\sqrt{vt}} ; \text{ Dimensionless} \quad (t = x/U_{\infty}) \quad \frac{y}{\sqrt{vt}} = \frac{CL}{\sqrt{[L^2][T^3]ET]} = \text{Dimensionless}$$

↳ For compressible, ν is not constant.

$$\therefore \nu = [L^2][T]^{-1}$$

- For compressible flow, the effect of density variable should be added on the definition of η .

$$\eta = \frac{y}{2\sqrt{vt}} = \frac{y}{2\sqrt{vX/U_{\infty}}} = \frac{y \rho}{2\sqrt{vX/U_{\infty}\rho^2}} = \frac{y \rho}{2\sqrt{\mu_p X/U_{\infty}}} = \frac{y \rho U_{\infty}}{2\sqrt{\mu_p X/U_{\infty} U_{\infty}^2}} = \frac{U_{\infty} \rho Y}{2\sqrt{\mu_p X U_{\infty}}}$$

Solutions of Compressible boundary layer Equations

↳ But Blasius solution is no longer valid.

- The mechanics of the transformation using the chain rule are similar to incompressible B.L. equations.

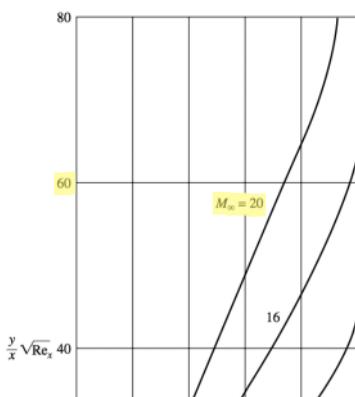
- A numerical self-similar solution can be obtained for the compressible and laminar flow over a flat plate.

↳ Note that the solution will depend on the Mach number, Prandtl number, and the condition of the wall.

- Indeed, the solution was obtained from Van Driest;

↳ zero heat transfer

- a) Velocity profiles in a compressible laminar boundary layer over an insulated flat plate



Observations.

$$\frac{\mu_w}{\mu_e} \approx \left(\frac{T_w}{T_e}\right)^{3/2} \frac{T_e + S}{T_w + S}$$

He used Sutherland's law for μ and assumed a constant $Pr = 0.75$

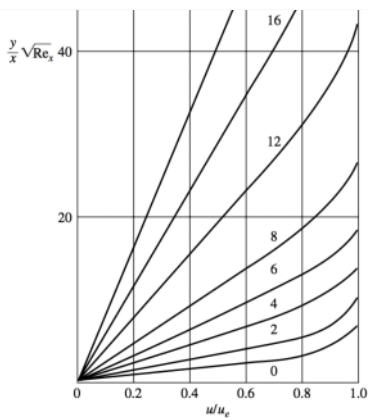
The B.L. thickness increases as $Mach$ is increased.

↳ In other words, B.L. thickness becomes large at large Mach number.

Then, why do we expect to have thicker B.L. as $Mach$ increases?

This would be because;
 $v \uparrow$ = High friction at the wall

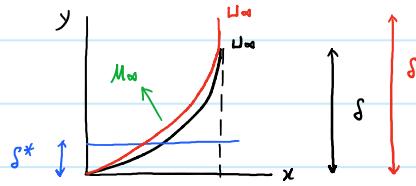
As $Mach$ increases, temperature would be increased within the boundary layer. ↗



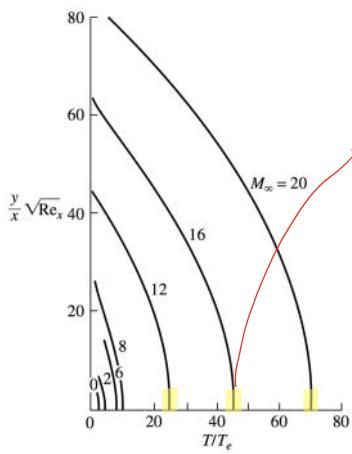
As M_∞ increases, temperature would be increased within the boundary layer. \uparrow

Since we assumed $p = \text{const.}$ in the Boundary layer equation, $p = \rho R T = \text{const.}$

$\rho \downarrow$ means thicker B.L. in order to satisfy the continuity. $m = \rho u A = \text{const.}$



b) Temperature profiles in a compressible laminar boundary layer over an insulated flat plate



Observations.

- As M_∞ increases to large, the temperatures increase.

- At the wall ($y = 0$), $\frac{dT}{dy}|_w = 0$ as it should be for an insulated surface ($q_w = 0$)

For cold T_w , $p = pRT$; assume $p = \text{const.}$ in BL.

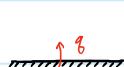
- higher ρ near wall
- more mass flow rate
- less momentum thickness
- reduction of displacement thickness

c) Velocity profiles in a laminar compressible boundary layer over a cold flat plate

→ It is now for the case of heat transfer to the wall. (cold wall : $T_w < T_{ad,w}$ vs. Hot wall : $T_w > T_{ad,w}$)

Types of the wall

(a) Isothermal wall



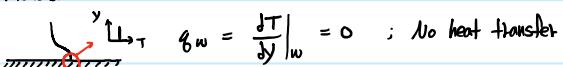
$$T_w = \text{const.} / \frac{dT_w}{dx} = 0$$

of. He assumed:

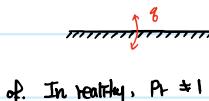
$$- T_w/T_e = 0.25$$

$$- Pr = 0.75$$

(b) Adiabatic wall

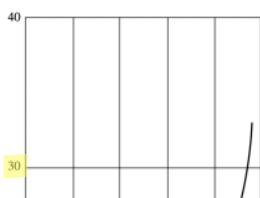


(c) "Real wall"



of. In reality, $Pr \approx 1$

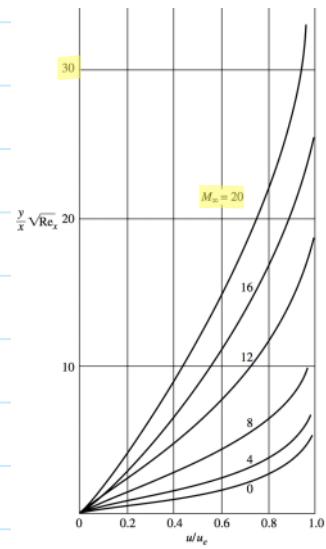
$$\left\{ \begin{array}{l} T_w = T_{ad,w} \text{ if } Pr = 1 \\ \text{Heat (Fluid to wall)} \text{ if } T_w < T_{ad,w} : \text{cold wall} \\ \text{Heat (wall to Fluid)} \text{ if } T_w > T_{ad,w} : \text{hot wall} \end{array} \right.$$



Observations.

- By comparing with the insulated wall, it is found that;

- The effect of a cold wall is to reduce the boundary thickness.



- The effect of a cold wall is to reduce the boundary thickness.

↳ Then, why?

: Because the pressure is the same in both cases, from $P = \rho R T$.

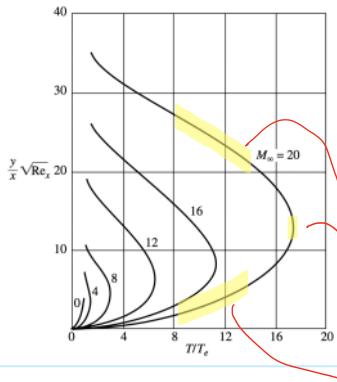
→ The density in the cold wall case is much higher.

↳ If the density is higher, we have to have smaller BL thickness.

(In order to maintain the mass flow, $m = \rho u A = \text{const.}$)

- Hence, the effect of a cold wall is to thin the boundary layer.

d) Temperature profiles in a laminar compressible boundary layer over a cold flat plate



Observations:

- By comparing with the insulated wall case, it is found that;
- The temperature levels of cold wall case are considerably lower than in the insulated case.
- Starting at the outer edge of the B.L. and going toward the wall, the temperature first increases.
- It reaches a peak somewhere within the boundary layer.
- ↳ The peak temperature inside the B.L. is an indication of the amount of viscous dissipation occurring within the B.L.
- And then, the temperature decreases to its pre-described cold wall value of T_w .

Adiabatic wall

For y -direction, we have

$$\rho \frac{D}{Dt} \left(\frac{1}{2} U_2^2 \right) = -U_2 \frac{\partial P}{\partial X_2} + U_2 \mu \frac{\partial^2 U_2}{\partial X_1 \partial X_2}$$

However, in this flow, $U_1^2 \ggg U_2^2$

$$\begin{aligned} \therefore \rho \frac{D}{Dt} \left(\frac{1}{2} U_1 U_1 \right) &\approx -U \frac{\partial P}{\partial X} + U \mu \frac{\partial^2 U}{\partial Y^2} \\ &= -U \frac{\partial P}{\partial X} + U \mu \frac{\partial^2 U}{\partial Y^2} + \cancel{\mu \left(\frac{\partial U}{\partial Y} \right)^2} - \cancel{\mu \left(\frac{\partial U}{\partial Y} \right)^2} \\ (\text{Now, we got an idea!}) \\ \text{This is what we've seen} &= -U \frac{\partial P}{\partial X} + \cancel{\mu \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right)} - \cancel{\mu \left(\frac{\partial U}{\partial Y} \right)^2} \end{aligned}$$

a) Adiabatic wall

(with the assumptions)

By looking back, we had a sort of Stagnation Enthalpy Equation.

$$\rho C_p \left(U \frac{\partial T_0}{\partial X} + V \frac{\partial T_0}{\partial Y} \right) = k \frac{\partial^2 T}{\partial Y^2} + \mu \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right)$$

If we look at the case of $P_r = 1$, we have

$$P_r = \frac{\mu C_p}{k} \Leftrightarrow k = \mu C_p \text{ if } P_r = 1$$

Then, the equation becomes;

$$\rho C_p \left(U \frac{\partial T_0}{\partial X} + V \frac{\partial T_0}{\partial Y} \right) = \mu C_p \frac{\partial^2 T}{\partial Y^2} + \mu \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right)$$

Also, if we consider $U \frac{du}{dy} = \frac{d}{dy} \left(\frac{1}{2} U^2 \right)$,

$$\rho C_p \left(U \frac{dT_0}{dx} + V \frac{dT_0}{dy} \right) = \mu \left(C_p \frac{\partial^2 T}{\partial y^2} + \frac{1}{2} \frac{\partial^2 U^2}{\partial y^2} \right)$$

$$= \mu C_p \frac{\partial^2}{\partial y^2} \left(T + \frac{U^2}{2C_p} \right)$$

$$* = \mu C_p \frac{\partial^2 T_0}{\partial y^2}$$

\therefore For $P_r = 1$, we have

$$\rho \left(U \frac{dT_0}{dx} + V \frac{dT_0}{dy} \right) = \mu \frac{\partial^2 T_0}{\partial y^2}$$

\Rightarrow we would say that the solution of this equation is

; Then, high temperature \Leftrightarrow low velocity $T_0 = \text{Constant}$ (only solution to satisfy the equation, both must be zero)

$$T_0 = T + \frac{1}{2} \rho U^2$$

* From the definition of the stagnation enthalpy,

$$h_0 = h + \frac{1}{2} U^2$$

$$\Leftrightarrow C_p T_0 = C_p T + \frac{1}{2} U^2$$

$$\Leftrightarrow T_0 = T + \frac{U^2}{2C_p} \quad V$$

$$= T + \frac{U^2}{a^2} \frac{a^2}{2C_p}$$

$$= T + M^2 \frac{(r-1)C_p T}{2C_p} \quad (\because a^2 = (r-1)C_p T)$$

Two types of BC in problem involving temperature distribution

1) Isothermal (P.D.E : Dirichlet B.C.)

- Known temperature at the wall

- Specify T_w and find heat transfer away from wall

2) Adiabatic (P.D.E. : Neumann B.C.)

- No heat transfer at the wall

- Find T_w

$$\begin{aligned} C_p &= \frac{r}{r-1} R \\ a^2 = rRT &\Leftrightarrow a^2 = x \cdot \frac{(r+1)C_p}{r} T \\ &= (r-1)C_p T \end{aligned}$$

So far, we have discussed it mathematically. So, what does it mean physically?

$$1 \quad a^2 = T \cdot R \Leftrightarrow a^2 = \frac{R}{C_p} T \\ = (1-1) C_p T$$

- So far, we have discussed it mathematically. So, what does it mean physically?

$$\cdot T_0 = T + \frac{U^2}{2C_p} = \text{const.}$$

In thermodynamics,

$$\Rightarrow \frac{dT}{dy} + \frac{1}{2C_p} \frac{dU^2}{dy} = 0$$

Adiabatic compression at wall,
where all K.E. converted into heat

$$\therefore T_w > T_\infty$$

- At wall, K.E. Completely converted into heat

This is because of $\rightarrow U=0 \rightarrow \frac{dT}{dy} = 0$; no heat transfer at wall
no-slip condition

- At freestream,

$$\frac{dU}{dy} = 0 \rightarrow \frac{dT}{dy} = 0 \text{; which means } T \rightarrow T_\infty$$

- This is a kind of "Adiabatic wall"

Some heat transfer
↑

- However, \rightarrow that means $T_w \neq T_{ad,w}$, then $\frac{dT}{dy} \neq 0$

: In reality, most of them aren't equal to $P_r = 1$.

- In order to fill in the gap in the analysis, the recovery

factor has been widely introduced.

Hence, not all of K.E. is converted
 \rightarrow we need to introduce Recovery factor.

$$\tau = \frac{T_{ad,w} - T_\infty}{T_{\infty,0} - T_\infty} < 1, \text{ usually } \tau \approx \sqrt{P_r} \text{ if } P_r < 1$$

$$\frac{\text{KE converted into heat}}{\text{KE available for conversion}} = \frac{C_p (T_{ad,w} - T_\infty)}{U_\infty^2 / 2} \text{ and divide by } C_p$$

(This was derived, see below)

$T_{ad,w} \uparrow \propto T_w \uparrow$

\therefore Fluid \rightarrow wall

(if $T_w < T_{ad,w}$)

- For instance,

$$T_w = T_{ad,w} \text{ if } P_r = 1$$

\Rightarrow Fluid

Wall \rightarrow fluid \rightarrow $T_w = T_{ad,w} = T_\infty$

wall

(This was derived, see below)

$$T_{ad,w} \uparrow \propto T_w \uparrow$$

: Fluid \rightarrow wall

(if $T_w < T_{ad,w}$)

Fluid

wall

• For instance,

$$T_w = T_{ad,w} \text{ if } \rho t = 1$$

Heat (Fluid to wall) if $T_w < T_{ad,w}$

Heat (wall to fluid) if $T_w > T_{ad,w}$

• In terms of Aerodynamic heating, it is more likely to occur

if the Mach number is high or if γ is relatively large ($\rho t \uparrow$)

: Here, Aerodynamic heating is defined as a heat flow from a cooler fluid to a warmer surface because of kinetic energy in the flow. It usually happens;

$$T_{ad,w} = T_w \left[1 + \frac{1}{2} (\gamma - 1) M_w^2 \right]$$

$$\Leftrightarrow T_{ad,w} - T_w = T_w \cdot \frac{1}{2} (\gamma - 1) M_w^2$$

Here, we knew that

$$T_{ad,w} = T_\infty + \gamma \left(\frac{T_w - T_\infty}{T_\infty} \right)$$

$T_\infty < T_w < T_{ad,w}$; In this condition, Aerodynamic heating occurs.

$$T_w = T_\infty \left[1 + \gamma \frac{1}{2} (\gamma - 1) M_w^2 \right] ; \text{ where } \gamma \approx \sqrt{\Pr}$$

Hence, (T_w)

Heat goes from cooler fluid

(T_w) to warmer body, as a

result, KE converted into

heat



Important issue for

design of surface for

thermal safety of

spacecraft.

e) Isothermal wall & Reynolds analogy

: This section will be covered in homework #7.

So, let me summarize the homework 7.

→ Next page

Transformation (2017 Fall)

• Idea from Blasius

- Used $g\theta$, $U = \frac{\partial \theta}{\partial y}$ and $V = -\frac{\partial \theta}{\partial x}$ ($\rho = \text{const}$) $\therefore \text{Result} = \frac{U}{U_\infty} = f'(y)$; where f = non-dimensional stream function

• For compressible BL,

- $\gamma \neq \text{const.}$ we need to use a different definition of $g\theta$ $\Rightarrow \rho U = \frac{d\theta}{dy}$ and $\rho V = -\frac{d\theta}{dx}$

For compressible BL,

$\rho \neq \text{const.}$, we need to use a different definition of $\lambda h \Rightarrow \rho u = \frac{dx}{dy}$ and $\rho v = -\frac{du}{dx}$

For continuity equation,

$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad ; \text{ automatically satisfied}$$

Ultimately, we would like to get : $\frac{u}{u_e} = \frac{df}{dy}$

$u = u_e f'$ from Blasius solution

$$\Rightarrow \rho u = \rho u_e f' \Leftrightarrow \frac{\partial f}{\partial y} = \rho u_e f'$$

$$\begin{aligned} u &= u_e f' && (\text{where } f \text{ denotes def.}) \\ \Rightarrow \rho u &= \rho u_e f' && \text{Integrate w.r.t. } y \\ \frac{\partial u}{\partial y} &= \rho u_e f' \rightarrow \frac{\partial^2 u}{\partial y^2} = \rho u_e f'' \rightarrow \frac{\partial^2 f}{\partial y^2} = \rho u_e f'' && \text{; where } e = \text{external to flow} \\ \text{Now,} & \quad \int_0^\infty \rho u_e f'' dy = (6.33) && \text{If } \rho, \mu, \text{ const. and } u_e \text{ const.} \\ \gamma &= \frac{u_e}{\sqrt{2} \rho} \int_0^y f'' dy = (6.34) && \gamma = \rho u_e u_x \\ \frac{\partial \gamma}{\partial y} &= \frac{u_e f''}{\sqrt{2} \rho \mu u_x} && \gamma = \frac{u_e f''}{\sqrt{2} \rho \mu u_x} \\ \frac{\partial \gamma}{\partial y} &= \frac{\rho u_e}{\sqrt{2} \rho \mu u_x} && \text{equiv to Blasius.} \end{aligned}$$

Finally,

Using the new definition of ε, γ , $\lambda h = \sqrt{2\varepsilon} f$

Then,

$$\rho u \frac{du}{dx} + \rho v \frac{du}{dy} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y} \left(\mu \frac{du}{dy} \right)$$

↓ Transformation

Equation 6.55 (Refer to the hand-out)

$$\begin{aligned} \text{Idea from Blasius not hand-in.} \\ \text{Using defn of } \varepsilon, \gamma, \psi = \sqrt{2\varepsilon} f \\ \text{transition BL analysis} \\ \text{from } \rho u \frac{du}{dx} + \rho v \frac{du}{dy} = -\frac{dp_e}{dx} + \frac{\partial}{\partial y} \left(\mu \frac{du}{dy} \right) \quad (6.28) \\ \text{to 6.55} \\ \frac{\partial}{\partial y} \left(C \frac{\partial f}{\partial y} \right) + f \frac{\partial^2 f}{\partial y^2} = \frac{2\varepsilon}{u_e} \left[\left(\frac{\partial f}{\partial y} \right)^2 - \frac{f''}{\rho} \right] \frac{du_e}{d\varepsilon} + 2\varepsilon \left(\frac{\partial f}{\partial y} \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial y \partial \eta^2} \right) \\ C = \frac{\rho u}{\rho_0 u_e} \quad (C=1 \text{ if } \rho, \mu \text{ are const.}) \\ \text{Chapman-Kubica parameter} \end{aligned}$$

Boundary conditions

$$\gamma = 0 \quad f = 0 \quad (\text{body surface is streamline if impermeable})$$

$$\left. \begin{array}{l} f' = 0 \quad (\text{no-slip}) \end{array} \right.$$

$$f' = 0 \quad (\text{no-slip})$$

$$g = g_w \quad (\text{if isothermal}) \quad \text{or} \quad g' = 0 \quad (\text{if adiabatic})$$

$$\gamma \rightarrow \infty : \quad f' = 1 \quad , \quad g' = 0$$

$$h \rightarrow \infty \quad , \quad (T \rightarrow T_{\infty})$$