Bernoulli's equation

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For the glory of God

Bernoulli's Equation

- Bernoulli's equation is probably the most famous equation in fluid dynamics.
- It presents the relationship between pressure and velocity in an inviscid and incompressible (In most cases) flow.
  - For compressible, we need to modify the general one.
- The physical significance of the equation is obvious from ; $p + \frac{1}{2}\rho u^2 = \text{const}$.
  - when the velocity increases, the pressure decreases.
- There is no requirement that the flow be irrotational for Bernoulli's equation to apply.

  The equation works fine in rotational flow.
  - In case of Rotational flow, $\therefore$ $p + \frac{1}{2}\rho u^2 = \text{const.}$ Along the streamline. Const. will change from one streamline to the next.

    : Bernoulli's equation is applicable along the streamline because Const. is not same for different streamlines.
  - In case of Irrotational flow, $\therefore$ $p + \frac{1}{2}\rho u^2 = \text{const.}$ throughout the flow

    : The equation is applicable at any given points because value of constant is same for all streamlines.

Derivation of the Bernoulli's Equation

- Bernoulli's equation was derived from the momentum equation.
- Assumptions : Inviscid, Incompressible, Steady (It is not necessary but it is used in most cases.), Negligible body force, perfect gas, Continuum

Momentum Equation : $\rho \frac{D\vec{u}}{Dt} = \cancel{\rho \vec{f}}^{\,0} - \nabla p + \cancel{\mu \nabla^2 \vec{u}}^{\,0} + \cancel{\frac{\mu}{3}\nabla\Delta}^{\,0}$; where $\Delta = \nabla \cdot \vec{u}$

$\Leftrightarrow \rho \frac{D\vec{u}}{Dt} = -\nabla p$; where $\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}$ (Substantial derivative)

For x-component $\Leftrightarrow \cancel{\frac{\partial u}{\partial t}}^{\,0} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

Hence, we have :

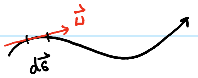
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Leftrightarrow u\frac{\partial u}{\partial x} + u\frac{dy}{dx}\frac{\partial u}{\partial y} + u\frac{dz}{dx}\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ; \text{ where } \begin{cases} u\,dz - w\,dx = 0 \\ v\,dx - u\,dy = 0 \end{cases}$$

from streamline equations

How can we obtain the Eqs ?

(see below or streamline equation note)



Here, $d\vec{s} \times \vec{u} = 0$

$(\because d\vec{s} \parallel \vec{u})$

Hence,

$$d\vec{s} \times \vec{u} = \begin{vmatrix} i & j & k \\ dx & dy & dz \\ u & v & w \end{vmatrix}$$

$= i(w\,dy - v\,dz)$

Multiply by dx, we have

$$u\left(\frac{\partial u}{\partial x}dx + \frac{dy}{\cancel{dx}}\frac{\partial u}{\partial y}\cancel{dx} + \frac{dz}{\cancel{dx}}\frac{\partial u}{\partial z}\cancel{dx}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$$\Leftrightarrow u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$$\Leftrightarrow u\,du = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$$\Leftrightarrow \frac{1}{2}d(u^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}dx$$

$\begin{cases} y\text{-component} : \frac{1}{2}d(v^2) = -\frac{1}{\rho}\frac{\partial p}{\partial y}dy \\ z\text{-component} : \frac{1}{2}d(w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial z}dz \end{cases}$

If we think about x, y, z components, and Add these equations ;

$$\frac{1}{2}d(u^2+v^2+w^2) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)$$

$$\frac{1}{2} d\left(u^2+v^2+w^2\right) = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)$$

$$\Leftrightarrow \quad \frac{1}{2} d(U^2) = -\frac{1}{\rho} dP$$

$$\Leftrightarrow \quad \frac{1}{2}\rho\, d(U^2) = -dP \quad \Leftrightarrow \quad dp = -\rho U dU$$

$$\begin{vmatrix} & & \\ u & v & w \\ & & \end{vmatrix}$$

$$= \hat{i}(w dy - v dz) - \hat{j}(w dx - u dz) + k(v dx - u dy) = 0$$

$$\therefore \; w dy - v dz = 0, \quad w dx - u dz = 0, \quad v dx - u dy = 0$$

By integrating, we have $1 \sim 2$ (It's going to be rotational form)

$$\frac{1}{2}\rho \int_1^2 d(U^2) = -\int_1^2 dP$$

$$\Leftrightarrow \quad \rho \int_1^2 U dU = -\int_1^2 dP \quad ; \; \rho = \text{constant}$$

$$\therefore \; \frac{1}{2}\rho\left(U_2^2 - U_1^2\right) = -\left(P_2 - P_1\right)$$

$$\rightarrow \; P_1 + \frac{1}{2}\rho U_1^2 = P_2 + \frac{1}{2}\rho U_2^2 \qquad \therefore \; P + \frac{1}{2}\rho U^2 = \text{const. along the streamline}$$

· What if we need to derive unsteady Bernoulli's Equation? : It will be directly related to irrotational form.

(Note that steady condition is not assumed at this time to obtain unsteady Bernoulli's Eq.)

· Then, the momentum equation becomes ;

$$\rho \frac{D\vec{U}}{Dt} = -\nabla P$$

$$\Leftrightarrow \quad \frac{d\vec{U}}{dt} + (\vec{U}\cdot\nabla)\vec{U} = -\nabla\frac{P}{\rho} \quad ; \text{ where } \rho = \text{const.}$$

vector identity

$$\equiv \frac{1}{2}\nabla(\vec{U}\cdot\vec{U}) - \vec{U}\times\underbrace{\cancel{(\nabla\times\vec{U})}}_{0\ (\because \text{ Irrotational})}$$

cf. $\vec{\omega} = \nabla\times\vec{U}$

Actually, here, we need to know ;

Irrotational → always non-viscous ∴ $\nu\nabla^2\vec{U} = 0$

$$\Leftrightarrow \quad \frac{d\vec{U}}{dt} + \frac{1}{2}\nabla(\vec{U}\cdot\vec{U}) = -\nabla\frac{P}{\rho}$$

$$\Leftrightarrow \quad \frac{d(\nabla\phi)}{dt} + \nabla\left(\frac{1}{2}U^2\right) = -\nabla\left(\frac{P}{\rho}\right) \quad ; \text{ where } \vec{U} = \nabla\phi \text{ for irrotational flow}$$

$$\Leftrightarrow \quad \nabla\left(\frac{d\phi}{dt}\right) + \nabla\left(\frac{1}{2}U^2\right) + \nabla\left(\frac{P}{\rho}\right) = 0$$

This is for irrotational flow.

(at any points)

· Hence, we have

$$\nabla\left(\frac{d\phi}{dt} + \frac{1}{2}U^2 + \frac{P}{\rho}\right) = 0 \quad \Leftrightarrow \quad \frac{d\phi}{dt} + \frac{1}{2}U^2 + \frac{P}{\rho} = \text{const.}$$

Known as unsteady Bernoulli's Equation

· If steady condition, $\frac{d}{dt} = 0$

$$\frac{1}{2}U^2 + \frac{P}{\rho} = \text{const} \quad \Leftrightarrow \quad P + \frac{1}{2}\rho U^2 = \text{const} \quad \text{(throughout the flow)}$$

· Is the Bernoulli's Equation valid for compressible flow? The answer is No!

↳ we need to modify the equation to obtain compressible form of

The strategy for solving problem in inviscid and incompressible flow the

The strategy for Solving problem in Inviscid and Incompressible flow

- obtain the velocity field from the governing Equation
- obtain the corresponding pressure field from Bernoulli's Equation

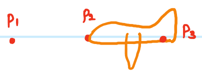
↳ we need to modify the Equation to obtain compressible form of Bernoulli's equation.

$$\boxed{\frac{u^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0}\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma} = \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0}}$$

cf. Bernoulli's Equation can be used to measure the airspeed of aircraft through pitot tube.

e.g.

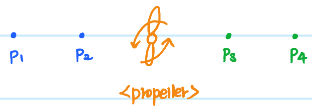
Under Inviscid assumption



: The Equation is only valid for P1 to P2

(P2 to P3 doesn't work due to viscous effects)

Also, please keep in mind it wouldn't be valid if there is a work done into the fluid.



- we can use the Bernoulli's before (P_1 and P_2) and after (P_3 and P_4).
- However, we cannot apply the Equation for P_2 and P_3.

↳ This is why there is a difference between P_2 and P_3 with respect to stagnation pressure.