

Reynolds analogy

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For the glory of God

In 2017, Professor Raffin asked " If the velocity profile is given, how would you calculate heat transfer into the wall ? "

- Reynolds analogy is popularly known to relate Laminar momentum and heat transfer.

- In boundary layer Equation,

$$U \frac{du}{dx} + V \frac{du}{dy} = V \frac{d^2 u}{dy^2} \quad (\text{x-momentum}) \quad \text{with assumptions : } \rho = \text{const flow}, \mu = \text{const}, \vec{f} = 0$$

steady, No pressure gradient

- Now let us think about the Enthalpy equation.

$$\rho \frac{dh}{dt} = \frac{DP}{Dt} + k \nabla^2 T + \mu \Xi \quad ; \text{ where } \Xi = 2S_{IJ}(S_{IJ} - \frac{1}{3} \Delta S_{IJ})$$

$k = \mu = \text{constants}$

$$\Leftrightarrow \rho C_p \frac{dT}{dt} = \frac{DP}{Dt} + k \nabla^2 T + \mu \Xi \quad \text{if calorically perfect gas}$$

$$\text{def. } S_{IJ} = \frac{1}{2} \left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} \right) = S_{JT}$$

$$\Leftrightarrow \rho C_p \left(\frac{\partial T}{\partial t} + U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} \right) = \frac{\partial P}{\partial t} + U \frac{\partial P}{\partial x} + V \frac{\partial P}{\partial y} + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu 2S_{IJ}(S_{IJ} - \frac{1}{3} \Delta S_{IJ}) \quad \text{for 2D laminar flow}$$

$$2(S_{IJ}^2 - \frac{1}{3} \Delta S_{IJ} S_{IJ}) \mu = 2\mu \left[S_{11}^2 + S_{12}^2 + S_{21}^2 + S_{22}^2 - \frac{1}{3} \Delta (S_{11}S_{11} + S_{12}S_{12} + S_{21}S_{21} + S_{22}S_{22}) \right]$$

$$= 2\mu \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left\{ \frac{1}{2} \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \right\}^2 + \left(\frac{\partial V}{\partial y} \right)^2 - \frac{1}{3} \left\{ \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \left(\frac{\partial U}{\partial x} \right) + \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) \left(\frac{\partial V}{\partial y} \right) \right\}^2 \right]$$

✓ due to steady assumption

✓ due to Boundary layer approximation equation

✓ due to No-slip condition with the flow over flat plate

✓ due to the fact that temperature remains mainly in y $\left| \frac{\partial T}{\partial y} \right| \ggg \left| \frac{\partial T}{\partial x} \right|$

- Hence, we have a simplified version of Enthalpy equation for 2D laminar flow with $\mu = k = \text{const.}$ and calorically perfect gas ($C_p = \text{const.}$).

$$\rho C_p \left(U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} \right) = U \frac{\partial P}{\partial x} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial U}{\partial y} \right)^2$$

- Now, let us think about the stagnation enthalpy equation ;

• Applying these simplification, we have

$$\rho C_p \left(u \frac{dT}{dx} + v \frac{dT}{dy} \right) = u \frac{dP}{dx} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{du}{dy} \right)^2$$

c) Stagnation Enthalpy Equation (sometimes, it's very useful)

Since $h_0 = h + \frac{1}{2} \vec{u} \cdot \vec{u}$, we might have

If $h_0 = \text{const.}$

high h (i.e. high T) occurs at location of minimum K.E.

$$\rho \frac{dh_0}{dt} = \rho \frac{dh}{dt} + \rho \frac{D}{Dt} \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right)$$

$$\Leftrightarrow \rho C_p \frac{DT_0}{Dt} = \rho C_p \frac{dT}{dt} + \rho \frac{D}{Dt} \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right)$$

; where h_0 = value of h at stagnation condition

(Sum of enthalpy and kinetic Energy)

• Again, if we think about the flow for Boundary Layer simplification,

$$\begin{aligned} \rho C_p \frac{DT_0}{Dt} &= \left\{ u \frac{dP}{dx} + k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{du}{dy} \right)^2 \right\} \\ &\quad + \left\{ -u \cancel{\frac{dP}{dx}} + \mu \frac{d}{dy} \left(u \frac{du}{dy} \right) - \mu \left(\frac{du}{dy} \right)^2 \right\} \dots \text{2nd term in RHS} \end{aligned}$$

$$\Leftrightarrow \rho C_p \frac{DT_0}{Dt} = k \frac{\partial^2 T}{\partial y^2} + \mu \frac{d}{dy} \left(u \frac{du}{dy} \right)$$

If we assume the flow is steady, we have

$$\rho C_p \left(u \frac{dT}{dx} + v \frac{dT}{dy} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \frac{d}{dy} \left(u \frac{du}{dy} \right)$$

By the way, you may be wondering where does the second term in Right hand side? ... 2nd term in RHS

: which is actually known as Kinetic Energy Equation in Boundary Layer Simplification.

Let's derive the equation (P.K. Young, office hour)

Starting from N-S Equation,

$$\rho \frac{DU_i}{Dt} = \cancel{\rho F_i} - \frac{\partial P}{\partial X_i} + \mu \frac{\partial^2 U_i}{\partial X_i \partial X_j} + \frac{\mu}{3} \frac{\partial \Delta}{\partial X_i}$$

$\rightarrow 0$ (\because Negligible)

If we consider the Boundary layer equations in 2D,

$$\Delta = \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx 0 \text{ (Very small)}$$

$$\rho \frac{DU_i}{Dt} = - \frac{\partial P}{\partial X_i} + \mu \frac{\partial^2 U_i}{\partial X_3 \partial X_5}$$

Also, for BL equations in x -direction,

Before looking at x -direction, let's arrange the equation more.

By dividing by ρ ,

$$\frac{D\bar{U}_T}{Dt} = - \frac{1}{\rho} \frac{\partial P}{\partial X_T} + \mu \frac{\partial^2 \bar{U}_T}{\partial X_5 \partial X_5} \frac{1}{\rho}$$

$$= - \frac{1}{\rho} \frac{\partial P}{\partial X_T} + U \frac{\partial^2 \bar{U}_T}{\partial X_5 \partial X_5}$$

Multiply U_T to each side,

$$U_T \frac{D\bar{U}_T}{Dt} = - \frac{U_T}{\rho} \frac{\partial P}{\partial X_T} + U_T U \frac{\partial^2 \bar{U}_T}{\partial X_5 \partial X_5}$$

Recall the Kinetic Energy Equation,

$$\frac{D}{Dt} \left(\frac{1}{2} U_T U_T \right) = U_T \frac{D\bar{U}_T}{Dt}$$

Hence, we would say that;

$$\frac{D}{Dt} \left(\frac{1}{2} U_T U_T \right) = - \frac{U_T}{\rho} \frac{\partial P}{\partial X_T} + U_T U \frac{\partial^2 \bar{U}_T}{\partial X_5 \partial X_5}$$

$$\Leftrightarrow \rho \frac{D}{Dt} \left(\frac{1}{2} U_T U_T \right) = - U_T \frac{\partial P}{\partial X_T} + U_T \mu \frac{\partial^2 \bar{U}_T}{\partial X_5 \partial X_5}$$

For x -direction, we have

$$\rho \frac{D}{Dt} \left(\frac{1}{2} U_1^2 \right) = - U_1 \frac{\partial P}{\partial X_1} + U_1 \mu \frac{\partial^2 U_1}{\partial X_2 \partial X_2}$$

$$\Leftrightarrow \rho \frac{D}{Dt} \left(\frac{1}{2} U^2 \right) = - U \frac{\partial P}{\partial X} + U \mu \frac{\partial^2 U}{\partial Y^2}$$

• For y -direction, we have

$$\rho \frac{D}{Dt} \left(\frac{1}{2} u_2^2 \right) = -u_2 \frac{\partial P}{\partial x_2} + u_2 \mu \frac{\partial^2 u_2}{\partial x_1 \partial x_2}$$

• However, in this flow, $u_1^2 \ggg u_2^2$

$$\begin{aligned} \therefore \rho \frac{D}{Dt} \left(\frac{1}{2} u_1 u_1 \right) &\approx -u \frac{\partial P}{\partial x} + u \mu \frac{\partial^2 u}{\partial y^2} && \text{Like mathematical trick} \\ &= -u \frac{\partial P}{\partial x} + u \mu \frac{\partial^2 u}{\partial y^2} + \cancel{u \left(\frac{\partial u}{\partial y} \right)^2} - \cancel{u \left(\frac{\partial u}{\partial y} \right)^2} \\ &= -u \frac{\partial P}{\partial x} + \underline{u \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right)} - \cancel{u \left(\frac{\partial u}{\partial y} \right)^2} \end{aligned}$$

(Now, we got an idea!)
This is what we've seen

d) Adiabatic wall

(with the assumptions)

• By looking back, we had a sort of Stagnation Enthalpy Equation.

$$\rho C_p \left(u \frac{\partial T_0}{\partial x} + v \frac{\partial T_0}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} + \mu \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right)$$

• If we look at the case of $P_r = 1$, we have

$$P_r = \frac{\mu C_p}{k} \Leftrightarrow k = \mu C_p \text{ if } P_r = 1$$

• Then, the equation becomes ;

$$\rho C_p \left(u \frac{\partial T_0}{\partial x} + v \frac{\partial T_0}{\partial y} \right) = \mu C_p \frac{\partial^2 T}{\partial y^2} + \mu \frac{\partial}{\partial y} \left(u \frac{\partial u}{\partial y} \right)$$

Also, if we consider $u \frac{du}{dy} = \frac{d}{dy} \left(\frac{1}{2} u^2 \right)$,

$$\rho C_p \left(u \frac{dT_0}{dx} + v \frac{dT_0}{dy} \right) = \mu \left(\frac{d^2 T}{dy^2} + \frac{1}{2} \frac{d^2 u^2}{dy^2} \right)$$

$$= \mu C_p \frac{d^2}{dy^2} \left(T + \frac{u^2}{2C_p} \right)$$

$$* = \mu C_p \frac{d^2 T_0}{dy^2}$$

\therefore For $P_r = 1$, we have

$$\rho \left(u \frac{dT_0}{dx} + v \frac{dT_0}{dy} \right) = \mu \frac{d^2 T_0}{dy^2} \Leftrightarrow u \frac{dT_0}{dx} + v \frac{dT_0}{dy} = v \frac{d^2 T_0}{dy^2}$$

\Rightarrow we would say that the solution of this equation is

$$T_0 = \text{constant}$$

* From the definition of the stagnation enthalpy,

$$h_0 = h + \frac{1}{2} u^2$$

$$\Leftrightarrow C_p T_0 = C_p T + \frac{1}{2} u^2$$

$$\Leftrightarrow T_0 = T + \frac{u^2}{2C_p} \quad v$$

$$= T + \frac{u^2}{a^2} \frac{a^2}{2C_p}$$

$$= T + M^2 \frac{(1-\gamma) C_p T}{2\gamma} \quad (\because a^2 = (1-\gamma) C_p T)$$

Finally, under the assumptions, we have

$$\text{momentum equation : } u \frac{du}{dx} + v \frac{du}{dy} = v \frac{d^2 u}{dy^2} \quad \text{for } x\text{-direction}$$

$$\text{stagnation energy equation : } u \frac{dT_0}{dx} + v \frac{dT_0}{dy} = v \frac{d^2 T_0}{dy^2}$$

Hence,

$$\rho \left(u \frac{dT_0}{dx} + v \frac{dT_0}{dy} \right) = \mu \frac{\partial^2 T_0}{\partial y^2} \dots (1)$$

- If we recall the momentum equation in Boundary layer,

$$u \frac{du}{dx} + v \frac{du}{dy} = \nu \frac{\partial^2 u}{\partial y^2} \dots (2)$$

- Divide (1) by ρ , we have

$$u \frac{dT_0}{dx} + v \frac{dT_0}{dy} = \nu \frac{\partial^2 T_0}{\partial y^2} \dots (3)$$

- Hence, if M_∞ is not high, we might be able to relate (2) to (3) and also (3) could be solved by Blasius Solution. (If the BC's are matched.)

This is an idea from Reynolds Analogy and he defined ; $\theta = \frac{T_0 - T_w}{T_{0,\infty} - T_w}$

$$\frac{u}{U_\infty} = \frac{T_0 - T_w}{T_{0,\infty} - T_w}$$

At $y=0$, $u=0$ and $T_0 = T_w$

At $y \rightarrow \infty$, $u = U_\infty$ and $T_0 = T_{0,\infty}$

(New Variable)

In other words,

$$\Leftrightarrow T_0 - T_w = \frac{u}{U_\infty} (T_{0,\infty} - T_w)$$

	momentum	Energy
At $y=0$	$\frac{u}{U_\infty} = 0$	$\frac{T_0 - T_w}{T_{0,\infty} - T_w} = 0$
At $y \rightarrow \infty$	$\frac{u}{U_\infty} = 1$	$\frac{T_0 - T_w}{T_{0,\infty} - T_w} = 1$

$$\Leftrightarrow T_0 = T_w + \frac{u}{U_\infty} (T_{0,\infty} - T_w)$$

- So, we got an idea where does it come from. Now, let's answer the question. (Make sure that this is only valid if $Pr=1$)

Answer



- In terms of Wall heat flux,

$$q_w = -k \frac{dT}{dy} \Big|_{y=0}$$

$$h_0 = h + \frac{1}{2} u^2$$

$$\Leftrightarrow q_p T_0 = c_p T + \frac{1}{2} k^2$$

$$\Leftrightarrow T_0 = T + \frac{1}{2c_p} u^2$$

- If we apply the Reynolds analogy to the equation,

$$q_w = -k \frac{d}{dy} \left(T_0 - \frac{u^2}{2c_p} \right) \Big|_{y=0}$$

$$= -k \frac{d}{dy} \left(T_w + \frac{u}{\text{Nu}} (T_{0,\infty} - T_w) - \frac{u^2}{2c_p} \right) \Big|_{y=0}$$

$$= -k \frac{d T_w}{dy} - k \frac{T_{0,\infty} - T_w}{\text{Nu}} \frac{du}{dy} \Big|_{y=0} + \frac{\alpha k}{2c_p} \left(u \frac{du}{dy} \right) \Big|_{y=0}$$

$\rightarrow 0$ (\because Isothermal, $T_w = \text{const}$) $\rightarrow 0$ ($\because u=0$
at wall)

$$\therefore q_w = - \frac{k (T_{0,\infty} - T_w)}{\text{Nu}} \frac{du}{dy} \Big|_{y=0}$$

- If we're trying to remember that $Z_w = u \frac{du}{dy} \Big|_{y=0}$

$$q_w = - \frac{k (T_{0,\infty} - T_w) Z_w}{\mu \text{Nu}}$$

- Now, let's think about the definition of Nusselt number.

$$\text{Nusselt number (Nu)} = \frac{|q_w|}{k \Delta T / L}$$

- Also, (MIT note) Reynolds analogy says :

- There is an approximate relation between skin friction coefficient and Stanton number ; $St = \frac{C_f}{2}$