Runge-Kutta method

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For the glory of God

Introduction

· In mathematics, an Ordinary Differential Equation (ODE) is defined as an equation containing an unknown function of one variable x and its derivative.

The unknown function is generally represented by a variable (often denoted y), which depends on x.

x is often called as the independent variable of the equation. (PDE if more than one independent variable)

(cf. Differential Equation is a mathematical equation that relates some function with its derivatives)

· For example,

$$\frac{dy}{dx} = \frac{x^2}{y} \quad ; \text{ This is an } \overset{1^{ST}\text{ order}}{\vee}\text{ ODE} \quad \text{vs.} \quad \frac{du}{dt} = \frac{d^2u}{dx^2} \quad ; \ 2^{ND} \text{ order PDE}$$

· Let us focus on ODE for this hand-written note.

: How would you solve the ODE described above?

· The first method that we can try is to solve the equation analytically. (e.g. Separation of variable)

$$\frac{dy}{dx} = \frac{x^2}{y} \Leftrightarrow y\,dy = x^2 dx$$

↳ By Integrating, we have

$$\int y\,dy = \int x^2 dx \Leftrightarrow \frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

$$\therefore y = \pm\sqrt{\frac{2}{3}x^3 + C}$$

· So, it seems that we were able to solve the ODE with respect to y analytically.

· Then, how about this ODE as below?

$$\frac{dy}{dt} = e^{-t^2} + y\ln(y)$$

: Would you be able to obtain analytic solution for this case?

↳ It might be difficult to solve it analytically.

· The second method that we can come up with is to use a numerical method such as Runge-Kutta.

· In this hand-written note, we will focus on Runge-Kutta method.

What is Runge-Kutta method ?

- Runge-Kutta method is known as a method to find numerical solution for Ordinary Differential Equation (ODE).
- The method was developed by the German mathematicians C. Runge and M.W. Kutta.
- The most widely known member of R-K family is generally referred to as RK4, fourth order Runge-Kutta method.

The fourth order Runge-Kutta method

- Let an Initial Value Problem (IVP) be specified as follows ;

$$\frac{dy}{dt} = f(t, y) \text{ and } y(t_0) = y_0 \quad ; \text{ where } y = \text{an unknown function that we want to approximate}$$

- Here, we assume that the function f, t_0, and y_0 are given.
- Also, in many practical applications, we consider that f is independent of t.
- In order to approximate the unknown function y, the RK4 proposes ;

1) $t_{n+1} = t_n + h$; where h is a step size
2) $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$; where $n = 0, 1, 2, 3, \cdots$

; Here, $k_1 = hf(t_n, y_n)$; The increment based on the slope at the beginning of the interval using y

$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$; The increment based on the slope at the midpoint of the interval using y, k_1

$k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$; The increment based on the slope at the midpoint of the interval using y, k_2

$k_4 = hf(t_n + h, y_n + k_3)$; The increment based on the slope at the end of the interval using y, k_3

- Hence, we will be able to estimate y_{n+1} using y_n, k_1, k_2, k_3, k_4, and h.

Example (Indian Institute of Technology)

- Find $y(1)$
- Given $h = 1.5$ and $y(0) = 5$, $x(0) = 0$
- ODE $dy/dx = 3e^{-x} - 0.4y$; then $f(x, y) = 3e^{-x} - 0.4y$

Solution)

- Using the fourth order Runge-Kutta method, $y(1)$ can be expressed as following ;

$$y(1) = y(0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

; where

$k_1 = hf(x_0, y_0) = (1.5)f(0,5) = 1.5\left(3e^{-0} - 0.4\times5\right) = 1.5$

$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = \text{Constant}$

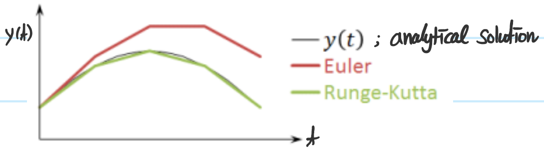
$\vdots$

- Therefore,

$x_1 = x_0 + h = 1.5$

$y_1 \simeq 4.365$

- Another example as below shows the accuracy between Euler and RK method w.r.t. analytical solution.



cf. Euler method (like 1^{st} order of RK)

$y_{n+1} = y_n + hf(x_n, y_n)$

Geometric reasoning of 4th order Runge-Kutta method

- Even if we did not derive the RK4 Equation formally (It's actually starting from Taylor Series Expansion, though), it is valuable to understand the geometric reasoning supporting RK4.
- The basic idea is to move from y_i to y_{i+1} by multiplying some estimated slope by a timestep.
- Let us discuss the components of RK4 Equation.

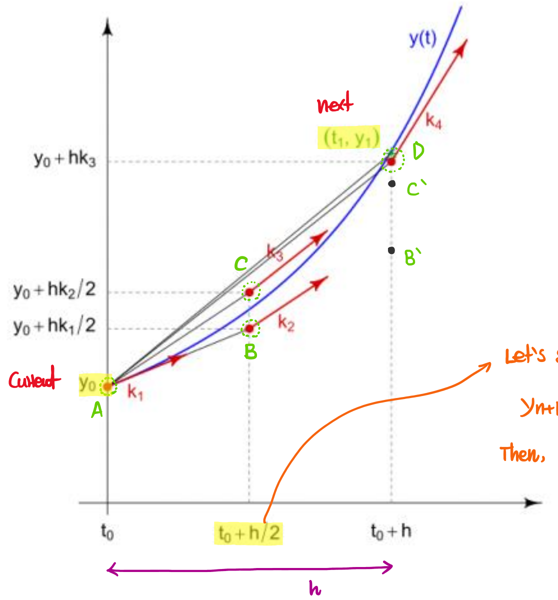
a) x iteration

- RK4 iterates the x values by simply adding a fixed step-size (h) at each iteration.

b) y iteration

- The y-iteration is far more interesting.
- It is a weighted average of four values k_1, k_2, k_3, and k_4.
  - ↳ Each of the k_i gives us an estimate of the size of the y-jump.

    (Eventually, we will get one really good slope)
- Let us take a look the plot as follows to get better understanding.



Let's say we have Euler Equation.

$y_{n+1} = y_n + h f(x_n, y_n)$

Then, when $x_n = t_0 + \frac{h}{2}$, y_n, (known)

we will have y_{n+1}.

⇒ Then, calculate the slope k_1

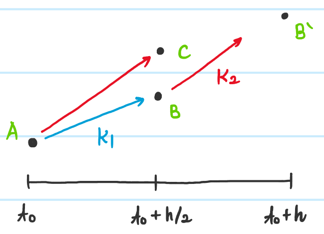
1) In terms of k_1

- k_1 is just the slope evaluated at the starting location y_i. (A)
- We project this slope forward by half a timestep ($t_0 + \frac{h}{2}$) to determine an estimated position. (B)
- This is simply Euler's method prediction.

2) In terms of k_2

- k_2 is the slope evaluated at the location B.
- We project this slope forward from the starting point (A) by half a timestep to determine an estimated position. (C)
- At this point, you may be confused what happen ..

↳ Let me walk through the two steps. (Given f function)



① Start with given information t_0, A, h, and $f(x_n, y_n)$

② Calculate B by $B = A + h f(t_0 + \frac{h}{2}, A)$

③ Calculate the slope k_1 between two points A and B

④ Calculate B` by $B` = B + h f(t_0 + h, B)$

⑤ Calculate the slope k_2 between two points B and B`

⑥ Calculate C by using the starting location and the slope k_2

(It means we project the slope k_2 from starting point A to the half a timestep.)

3) In terms of k_3

· k_3 is the slope evaluated at the location C.

· This is quite similar to that of k_2 except that where k_1 used to be, there is now k_2.

· We project this slope forward from the starting point by a full timestep to determine an estimated position. (D)

- In similar way, { first, we calculate C' based on given information.
  Second, we calculate k_3 slope between C and C'.
  Lastly, we project the slope k_3 from starting point to a full timestep.

4) In terms of k_4

· k_4 is the slope evaluated at the location D.

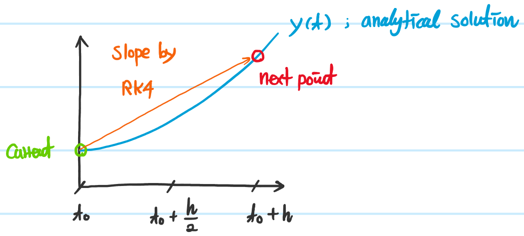
· We project this slope forward from the starting point by a full timestep.

↳ Finally, we will arrive to the next point.

c) Next position estimation

· Once we have these four slopes, we can combine them in a weighted fashion to create an average slope and use the average slope to compute the next position.

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$



cf. RK4 numerical solution may be varied as a step size changes.