Supersonic thin airfoil theory

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For the glory of God

Introduction

- The calculation of lift and drag for an airfoil at supersonic speed is different to lower speed airfoils.

  ↳ This is because the physics of a supersonic flow is completely different from that of a subsonic flow.

- The linearized full potential equation was derived in chapter 11 (Anderson);

$$(1-M_\infty^2)\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0$$

  - In chapter 11, we treated the equation with subsonic assumption where $1-M_\infty^2>0$
  - However, the equation holds for both subsonic and supersonic flow where $1-M_\infty^2<0$

- If supersonic flow, it seems to be a change in sign on the first term, which results in a dramatic change in reality.

(Mathematically) for example $\begin{cases} \text{if } 1-M_\infty^2>0, \text{ Equation becomes Elliptic P.D.E.} \\ \text{otherwise } 1-M_\infty^2<0, \text{ Equation becomes Hyperbolic P.D.E.}\end{cases}$ ∴ There is a difference

↳ For more details, please see Method of characteristics note

Derivation of the linearized supersonic pressure coefficient formula

- For the case of supersonic flow,

$$(1-M_\infty^2)\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0 \underset{\text{Multiply by }-1}{\Leftrightarrow} (M_\infty^2-1)\frac{\partial^2\hat{\phi}}{\partial x^2}-\frac{\partial^2\hat{\phi}}{\partial y^2}=0 \quad ; \text{ where let } \lambda \equiv \sqrt{M_\infty^2-1}$$

Therefore, we have

$$\lambda^2\frac{\partial^2\hat{\phi}}{\partial x^2}-\frac{\partial^2\hat{\phi}}{\partial y^2}=0$$

- Then, what is a solution of the equation?

⇒ Ackeret suggested the solution in his paper; $\hat{\phi}=f(x-\lambda y)$

- How do we know the functional relation $\hat{\phi}=f(x-\lambda y)$ is a solution of the governing equation?

∵ We can demonstrate this by substitution each term such as; (let $\hat{\phi}(x,y)=F(\xi)$ where $\xi=x-\lambda y$)

a) $\frac{\partial^2\hat{\phi}}{\partial x^2}$ term

$$\frac{\partial\hat{\phi}}{\partial x}=\frac{dF}{d\xi}\frac{\partial\xi}{\partial x}=F'(\xi)\cdot 1=\frac{dF}{d\xi}$$

$$\frac{\partial^2\hat{\phi}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\hat{\phi}}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{dF}{d\xi}\right)=\frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial x}\left(\frac{dF}{d\xi}\right)=\frac{d^2F}{d\xi^2}$$

b) $\dfrac{\partial^2 \hat{\phi}}{\partial y^2}$ term

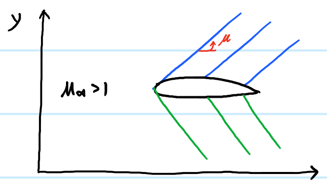
$$\frac{\partial \hat{\phi}}{\partial y} = \frac{dF}{dy} = \frac{dF}{d\xi}\frac{d\xi}{dy} = -\lambda \frac{dF}{d\xi}$$

$$\frac{\partial^2 \hat{\phi}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \hat{\phi}}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\lambda \frac{dF}{d\xi}\right) = \frac{d}{d\xi}\frac{d\xi}{dy}\left(-\lambda\frac{dF}{d\xi}\right) = \lambda^2 \frac{dF}{d\xi}$$

Hence, by substituting each term into the equation, we have

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0 \iff \lambda^2 \frac{d^2F}{d\xi^2} - \lambda^2 \frac{d^2F}{d\xi^2} = 0 \quad \therefore \text{Satisfied.}$$

· In the same way, we will end up realizing $\phi = g(x+\lambda y)$ also satisfies the governing equation.



– : $f(x-\lambda y) = \text{const.}$

– : $g(x+\lambda y) = \text{const.}$

Disturbances propagate everywhere.

Note that ;

the Mach waves slope downstream only above the wall. Thus, any disturbance at the wall cannot propagate upstream

→ Major difference to subsonic flow

· Now, let's examine the functional relations such as $f(x-\lambda y)$

⇒ Even though f is not very specific, the f tells us something specific about the flow, namely,

↳ This is because f can be any function of $x-\lambda y$.

$\hat{\phi}$ is constant along lines of $x-\lambda y = \text{constant}$

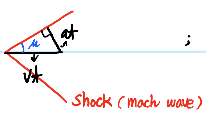
($\because$ if $x-\lambda y = \text{const}$, f is constant $\iff \hat{\phi}$ is constant, $\hat{\phi} = f(x-\lambda y)$)

⇒ Then, $x-\lambda y = \text{const} \iff \dfrac{d}{dx}(x-\lambda y) = \dfrac{d}{dx}(\text{const})$

$$\iff 1 - \lambda \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

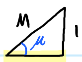
; this was also derived in M.O.C. note with a different way.

· Here, if we can remember correctly,



; $\sin\mu = \dfrac{at}{Vt} = \dfrac{1}{M}$ $\iff$

Mach angle



$\therefore \sin\mu = \dfrac{1}{M}$

↳ It should be $\sqrt{M^2-1}$

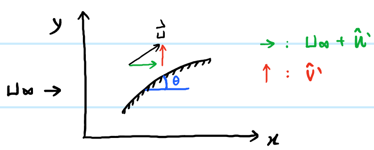
In linearized supersonic flow, $\therefore \tan\mu = \dfrac{1}{\sqrt{M^2-1}}$

· By comparing the results, it is found that : information is propagated along Mach lines where the Mach angle $\mu = \sin^{-1}\left(\frac{1}{M_\infty}\right)$

– A line along which $\hat{\phi}$ is constant is a Mach line. (The slope $\frac{1}{\lambda}$ is directly related to the freestream Mach angle)

· As a final step, let's think about the boundary condition for linearized small perturbations flow.

– Let $\theta \equiv$ Surface inclination angle relative to freestream direction



→ : $U_\infty + \hat{u}'$

↑ : $\hat{v}'$

; where $\hat{v}' = (U_\infty + \hat{u}')\tan\theta$

$\simeq U_\infty \theta$ for small angle and $U_\infty >>> \hat{u}'$

– Here, we know that ;

- Here, we know that ;

$$\hat{u} = \frac{\partial \hat{\phi}}{\partial x} = F'(\xi) \qquad \Leftrightarrow \quad \hat{u} = -\frac{\hat{v}}{\lambda} \simeq -\frac{U_\infty \theta}{\lambda}$$

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = -\lambda F'(\xi)$$

- Recall the linearized pressure coefficient, (from Anderson)

$$C_p = -\frac{2\hat{u}}{U_\infty} = \frac{2\theta}{\lambda} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$ ∴ It is the linearized supersonic pressure coefficient

↳ It states that C_p is directly proportional to the local surface inclination angle w.r.t. the freestream.

(It holds for any slender two-dimensional body where θ is small)

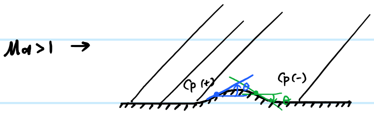
· We must be careful with signs for the equation ;

- Remember θ is positive that give $C_p > 0$ where the surface is inclined toward the freestream.

e.g. $M_\infty \rightarrow$ $C_p > 0$ or $M_\infty \rightarrow$ $C_p > 0$

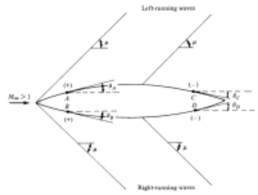
- When θ is negative, it results in $C_p < 0$ where the surface is inclined away from the freestream.

e.g. $M_\infty \rightarrow$ $C_p < 0$ or $M_\infty \rightarrow$ $C_p < 0$



$\theta > 0$ ∴ $C_p > 0$

$\theta < 0$ ∴ $C_p < 0$



⇒ It is interesting that this theory also predicts a finite wave drag although shock waves themselves are not treated in such linearized theory.

· Finally, we have ;

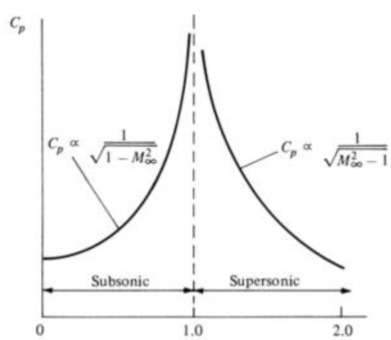


Figure 12.2 Variation of the linearized pressure coefficient with Mach number (schematic).

· For supersonic flow,

- C_p decreases as M_∞ increases.

· For subsonic flow,

- C_p increases as M_∞ increases.

· Note that

- Both results predict $C_p \rightarrow \infty$ as $M_\infty \rightarrow 1$

- Neither supersonic nor subsonic is valid in the transonic range around Mach 1.

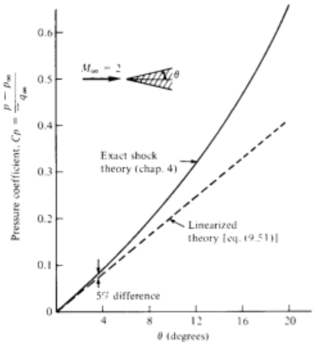
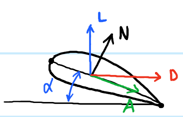


Figure 9.8 Comparison between linearized theory and exact shock results for the pressure on a wedge in supersonic flow.

Application to supersonic flat plate

· With the distribution of C_p over the airfoil surface, the lift and drag can be obtained from the integrals of the equation.

$U_\infty \rightarrow$



$$\text{Lift} = N\cos\alpha - A\sin\alpha$$

$$\text{Drag} = N\sin\alpha + A\cos\alpha$$

; where C_n is a coefficient of Normal force (N)

↳ function of C_p and C_f

· Let us consider the simplest possible airfoil, namely, a flat plate at a small angle of attack (α)

→ function of C_p and C_f

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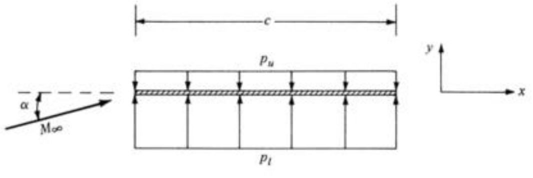
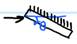
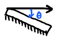


Figure 12.4 A flat plate at angle of attack in a supersonic flow.

$$C_{p,\,lower} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$ $U_\infty \rightarrow$  $\theta > 0$

$$C_{p,\,upper} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$ $U_\infty \rightarrow$  $\theta < 0$

- Since the surface inclination angle is constant along the entire lower surface, C_p has a constant value over the surfaces.

$$C_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\,dx$$; where C_f term canceled out ($\because$ Inviscid)

$$C_a = \frac{1}{c}\int_0^c (C_{p,u} - C_{p,l})\,dy \simeq 0$$ ($\because$ the flat plate has theoretically zero thickness, so that $dy = 0 \rightarrow C_a = 0$)

- Then, finally,

$$C_l = C_n \cos\alpha - C_a \cancel{\sin\alpha} \quad (\rightarrow 0)$$

$$\simeq C_n \quad \text{for small } \alpha \text{ assumption}$$

$$\therefore C_l = C_n$$

$$= \frac{1}{c}\int_0^c \left\{ \frac{2\alpha}{\sqrt{M_\infty^2-1}} - \left(-\frac{2\alpha}{\sqrt{M_\infty^2-1}}\right) \right\} dx$$

$$= \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$; C_l depends on only α within the approximation of linearized theory.

· In terms of drag coefficient,

- Recall that, for incompressible flow, $C_d = 0$ in the absence of viscosity.

- However, the linearized theory for supersonic flow provides a wave-drag coefficient.

(Keep in mind that the result is only valid for small α) ↳ Drag due to compressibility

$$C_d = C_n \sin\alpha + \cancel{C_a}\cos\alpha$$ ↳ 0 ($\because$ No thickness for thin flat plate, theoretically)

$$\simeq C_n\,\alpha \quad (\because \sin\alpha \simeq \alpha \text{ for small } \alpha)$$

$$\therefore C_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}$$ → Although we have neglected viscosity, we still have drag.

What if we have an airfoil with thickness and camber at angle of attack?

→ The wave-drag coefficient would be as follows ;

$$C_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + g_c^2 + g_t^2\right)$$; where g_c and g_t are functions of the airfoil camber and thickness

$$= C_{d,\,lift} + C_{d,\,t\&c}$$

$$= C_{d,\,lift} + C_{d,\,t\&c}$$

wave drag due to lift

wave drag due to thickness and camber

(Busemann's theory) → He got 2ND order with respect to these terms