

N-S equations

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For the glory of God

This question was made by me to prepare Aero qualifying exam. This would be one of popular topics for qual. indeed.

Question) Write down Navier - Stokes Equations.

Answer)

Continuity Equation : $\frac{\partial \vec{u}}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$

$$\Leftrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Momentum Equation : $\rho \frac{D\vec{u}}{Dt} = \rho \vec{f} - \nabla p + \mu \nabla^2 \vec{u} + \frac{\mu}{3} \nabla \Delta$; where $\Delta = \nabla \cdot \vec{u}$

$$\Leftrightarrow x\text{-momentum} : \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho f_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \frac{\mu}{3} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\Leftrightarrow y\text{-momentum} : \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho f_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \frac{\mu}{3} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\Leftrightarrow z\text{-momentum} : \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho f_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \frac{\mu}{3} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Energy Equation : $\rho \frac{D}{Dt} (e + \frac{1}{2} \vec{u} \cdot \vec{u}) = \rho \vec{f} \cdot \vec{u} + \nabla \cdot (Z_{IJ} \vec{u}) + \nabla \cdot (k \nabla T) \rightarrow$ Enthalpy Equation : $\rho \frac{Dh}{Dt} = \frac{\partial p}{\partial t} + \nabla \cdot (k \nabla T) + \mu \Xi$

\downarrow convection \downarrow compressibility \downarrow conduction \downarrow viscous dissipation

Question) Write them in Tensor notation.

$$; \text{where } \Xi = 2 S_{IJ} (S_{IJ} - \frac{1}{3} \Delta g_{IJ})$$

Answer) Continuity : $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_I} (\rho u_I) = 0$

Momentum : $\rho \frac{D u_I}{Dt} = \rho f_I - \frac{\partial p}{\partial x_I} + \mu \frac{\partial^2 u_I}{\partial x_J \partial x_K} + \frac{\mu}{3} \frac{\partial}{\partial x_I} \left(\frac{\partial u_I}{\partial x_I} \right)$

Energy : $\rho \frac{D}{Dt} (e + \frac{1}{2} u_I u_I) = \rho f_I u_I + \frac{1}{\partial x_I} (Z_{IJ} u_I) + \frac{1}{\partial x_I} (k \frac{\partial T}{\partial x_I})$

Question) What is a physical meaning of them?

Answer)

Continuity Equation : Mass conservation = Mass can be neither created nor destroyed.

Momentum equation : It is an application of Newton's 2nd law of motion to an element of the fluid.

$$\hookrightarrow \text{Force} = \text{Time rate of change of momentum} \quad (\vec{F} = \frac{d}{dt} (m \vec{u}))$$

Energy Equation : It is an application of the 1st law of thermodynamics to a system.

$$\hookrightarrow \text{Energy can be neither created nor destroyed but it can only change in form.}$$

$$(de = \delta q - \delta W)$$

Question) Between Eulerian and Lagrangian methods, which one would you choose to derive NS equations? and how come?

Answer)

Eulerian method is more preferred than Lagrangian method for the following reasons :

- It is often extremely difficult to identify and follow the same mass of fluid at all times

as must be done in Lagrangian method.

- Our primary interest is often not in the motion of a given mass of fluid but rather in the effect of the overall fluid motion

(Question) Let's say you ended up choosing Eulerian method to derive NS equations. If so, Reynolds transport theorem must be introduced. why?

Answer)

- Since physical laws apply to systems (not to control volume), we need some sort of a mechanism to somehow transform the law to its equivalent control volume formulation.

Hence, this theorem relates change in properties of a system to change in properties in a control volume.

→ In other words, the theorem relates derivatives in the Lagrangian framework to derivatives in the Euler one.

(Question) Using the theorem, derive integral form of continuity equation.

Answer)

- In terms of mass conservation, the theorem says:

$$\frac{dM}{dt} = \frac{\partial}{\partial t} \iiint_{cv} \rho dV + \iint_{cs} \rho \vec{v} \cdot d\vec{s} \quad (\text{Lagrangian framework} \leftrightarrow \text{Eulerian framework})$$

⇒ Total rate of change of Mass in the fluid system

= Rate of change of mass inside control volume + Net mass flow rate out of control volume through surface S

- Based on the mass conservation, $\frac{dM}{dt} = 0$

$$\frac{\partial}{\partial t} \iiint_{cv} \rho dV + \iint_{cs} \rho \vec{v} \cdot d\vec{s} = 0 ; \text{ Integral form of continuity equation}$$

(Question) Using divergence theorem, derive differential form of continuity equation

Answer)

- By using Divergence theorem,

$$\text{LHS} : \iint_s \rho \vec{v} \cdot d\vec{s} \Rightarrow \iiint_v \nabla \cdot \rho \vec{v} dV$$

- Also, since the control volume is fixed in time from the derivation process, we have

$$\text{RHS} : \frac{\partial}{\partial t} \iiint_v \rho dV \Rightarrow \iiint_v \frac{\partial \rho}{\partial t} dV$$

- Hence, we have

$$\iiint_v \nabla \cdot \rho \vec{v} dV + \iiint_v \frac{\partial \rho}{\partial t} dV = 0$$

$$\Rightarrow \iiint_v \left(\nabla \cdot \rho \vec{v} + \frac{\partial \rho}{\partial t} \right) dV = 0 ; \text{ let cv shrink to a point, then inside must be zero.}$$

↓ Then, obtain a PDE

$$\therefore \frac{dp}{dt} + \nabla \cdot (\rho \vec{u}) = 0 \rightarrow \text{Differential form of the Continuity Equation}$$

Question) Using substantial derivative, write the continuity equation with substantial derivative

Answer)

The continuity equation can be also written with substantial derivative. $\frac{D(\rho)}{Dt} = \frac{d(\rho)}{dt} + \vec{u} \cdot \nabla(\rho)$

$$\frac{dp}{dt} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\Leftrightarrow \frac{dp}{dt} + \frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) + \frac{d}{dz}(\rho w) = 0 \quad \text{and convective derivative.}$$

$$\Leftrightarrow \frac{dp}{dt} + \rho \frac{\partial u}{\partial x} + u \frac{\partial p}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial p}{\partial y} + \rho \frac{\partial w}{\partial z} + w \frac{\partial p}{\partial z} = 0$$

$$\Leftrightarrow \frac{dp}{dt} + \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla p = 0$$

$$\Leftrightarrow \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \rightarrow \text{Continuity Equation with Substantial derivative}$$

It is about the time rate of change following a moving fluid element. It is consisting of both local derivative

\downarrow

physically the time rate of change at a fixed point, $\frac{d(\rho)}{dt}$

physically the time rate of change due to the movement of the fluid element from one location to another in the flow field, $\nabla \cdot \vec{u}()$

Question) Derive momentum Equation (please do not expand surface stress)

Answer)

We are still considering the Eulerian method to derive the equation.

According to the Newton's 2nd law on the system,

$$\text{Force} = \text{Total Time rate of change of momentum}$$

where

$$\text{Body force} = \iiint_{cv} \rho \vec{f} dV$$

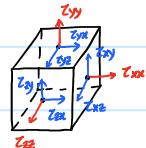
LHS : Force \rightarrow Body force and Surface force

RHS : $\frac{d(m\vec{v})}{dt} = \frac{d}{dt} \iiint_{cv} \rho \vec{u} dV + \iint_{cs} (\rho \vec{u} \cdot \vec{ds}) \vec{u}$; Reynolds transport theorem

\downarrow

\circ (momentum conservation)

$$\text{Surface force} = \iint_{cs} \tau_{ij} d\vec{s} \quad (Z = \frac{F}{A} \leftrightarrow F = ZA) \quad \text{Hence, Total Time rate of change of momentum is :}$$



$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Time rate of change of momentum of fluid inside CV

Net flow of momentum out of CV across CS

Therefore,

Force = total time rate of change of momentum

$$\Leftrightarrow \iiint_{cv} \rho \vec{f} dV + \iint_{cs} \tau_{ij} d\vec{s} = \frac{d}{dt} \iiint_{cv} \rho \vec{u} dV + \iint_{cs} (\rho \vec{u} \cdot \vec{ds}) \vec{u}$$

By using the Divergence theorem, we have

$$\iint_{cs} \tau_{ij} d\vec{s} = \iiint_{cv} \nabla \cdot \tau_{ij} dV$$

$\iint_{cs} (\rho \vec{u} \cdot \vec{ds}) \vec{u} = \iiint_{cv} \nabla \cdot (\rho \vec{u} \vec{u}) dV$ Note that this is a scalar

Hence,

$$\iiint_V \frac{\partial}{\partial t} \rho dV + \iiint_V (\nabla \cdot \vec{Z}_{IJ}) dV = \iiint_V \frac{1}{dt} (\rho \vec{U}) dV + \iiint_V \nabla \cdot (\rho \vec{U} \vec{U}) \vec{U} dV$$

$$\iiint_V \left\{ \frac{d(\rho \vec{U})}{dt} + \nabla \cdot (\rho \vec{U} \vec{U}) - \rho \vec{F} - \nabla \cdot \vec{Z}_{IJ} \right\} dV = 0$$

$$\therefore \frac{d(\rho \vec{U})}{dt} + \nabla \cdot (\rho \vec{U} \vec{U}) = \rho \vec{F} + \nabla \cdot \vec{Z}_{IJ}$$

$$\Leftrightarrow \rho \frac{D\vec{U}}{Dt} = \rho \vec{F} + \nabla \cdot \vec{Z}_{IJ} \quad \text{orange arrow} \quad \frac{d(\rho \vec{U})}{dt} + \nabla \cdot (\rho \vec{U} \vec{U}) = \rho \frac{D\vec{U}}{Dt} + \vec{U} \frac{d\rho}{dt} + \rho \vec{U} \nabla \cdot \vec{U} + \rho (\vec{U} \cdot \nabla) \vec{U} + \vec{U} (\vec{U} \cdot \nabla) \rho$$

Here, with tensor format, $\rho \frac{D\vec{U}_T}{Dt} = \rho \vec{F}_T + \frac{\partial Z_{TT}}{\partial X_3}$

$$= \rho \left(\frac{D\vec{U}}{Dt} + (\vec{U} \cdot \nabla) \vec{U} \right) + \vec{U} \left(\frac{d\rho}{dt} + \rho \nabla \cdot \vec{U} + \vec{U} \cdot \nabla \rho \right)$$

$$= \rho \frac{D\vec{U}}{Dt}$$

$$\rightarrow 0 \quad (\because \text{Continuity Eq.})$$

Question) Derive momentum equations with Stokes hypothesis

Answer)

- First of all, Stokes postulated the surface stress as following to build up relationship between surface stress and velocity.

$$Z_{IJ} = \text{normal stress} + \text{shear stress}$$

$$= -P \delta_{IJ} + \sigma_{IJ} ; \text{ where Newtonian fluid with } \mu = \text{const}$$

$$= -P \delta_{IJ} + \mu \left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} \right) + \lambda (\nabla \cdot \vec{U}) \delta_{IJ} ; \text{ where } \lambda = -\frac{2}{3} \mu \quad (1: \text{Second viscosity})$$

①

②

③

④

For ①,

- pressure is the same in all directions (isotropic).
- In terms of normal stress, it only consider \vec{U} this direction.

For ②,

- In terms of shear stress, no rotation rate acts ($\tau_{ij} = 0$) because the flow is assumed to be stationary.
- So, only shear rate exists.

$$S_{IJ} = S_{JI} = \frac{1}{2} \left(\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} \right) \quad \text{i.e. } S_{11} = \frac{\partial u}{\partial x} \text{ and } S_{12} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = S_{21}$$

For ③,

- In incompressible flow, this term is so small that we can neglect the term.
- As velocity gradient becomes extremely large on the face of fluid element, there can be a meaningful viscous-induced normal force on each face, which acts in addition to the pressure.

- Hence, normal stress $\propto \lambda$ diagonal term of velocity gradient ; where $\lambda = \text{secondary viscosity}$

For ④,

- Thermodynamic pressure is most likely a pressure at thermodynamic equilibrium.
- Stokes introduced the mechanical pressure because a flowing fluid is not in thermodynamic equilibrium.

- He postulated that $\bar{Z}_{IJ} = -P\delta_{IJ} + \delta_{IJ}$ becomes $Z_{II} = -P\delta_{II} + \delta_{II}$ when the fluid is at rest.

$$\left. \begin{array}{l} \text{For inviscid, } Z_{II} = -P\delta_{II} (\because \delta_{II} = 0 \text{ for inviscid flow}) \\ \text{For viscous, } Z_{II} = -P\delta_{II} (\because \delta_{II} = 0 \text{ from Stokes hypothesis}) \end{array} \right\} \quad \therefore P = -\frac{1}{3} Z_{II}$$

$\delta_{II} = 1+1+1=3$

- He defined a Mechanical pressure as : $\bar{P} = \text{thermodynamic pressure} + \text{a component induced by stress due to the motion of fluid}$

$$\begin{aligned} \bar{P} &\equiv -\frac{1}{3} Z_{II} \\ &= -\frac{1}{3} (-P\delta_{II} + \delta_{II}) \\ &= -\frac{1}{3} (-3P + \mu \left(\frac{\partial u_I}{\partial x_I} + \frac{\partial u_I}{\partial x_I} \right) + 3\lambda (\nabla \cdot \vec{u})) \\ &= P - \frac{2}{3}\mu (\nabla \cdot \vec{u}) - \lambda (\nabla \cdot \vec{u}) \\ &= P - (\lambda + \frac{2}{3}\mu) \nabla \cdot \vec{u} \end{aligned}$$

- He thought that it would be easy if it is assumed that "Mechanical pressure = Thermodynamic pressure"

$$\left. \begin{array}{l} \text{For incompressible, } \bar{P} = P (\because \nabla \cdot \vec{u} = 0) \\ \text{For compressible, } \bar{P} = P (\because \lambda + \frac{2}{3}\mu = 0 \Leftrightarrow \lambda = -\frac{2}{3}\mu, \text{ which is Stokes hypothesis}) \end{array} \right\} \quad \text{It's still controversial, though.}$$

- Let's go back to the equation :

$$\begin{aligned} Z_{IJ} &= \text{Normal stress} + \text{Shear stress} \\ &= -P\delta_{IJ} + \delta_{IJ} ; \text{ where Newtonian fluid with } \mu = \text{const} \\ &= -P\delta_{IJ} + \mu \left(\frac{\partial u_I}{\partial x_J} + \frac{\partial u_J}{\partial x_I} \right) + \lambda (\nabla \cdot \vec{u}) \delta_{IJ} ; \text{ where } \lambda = -\frac{2}{3}\mu \quad (\lambda : \text{Second viscosity}) \\ &= -P\delta_{IJ} + 2\mu S_{IJ} + \left(-\frac{2}{3}\mu \right) \Delta \delta_{IJ} ; \text{ where } S_{IJ} = \frac{1}{2} \left(\frac{\partial u_I}{\partial x_J} + \frac{\partial u_J}{\partial x_I} \right) \text{ and } \Delta = \nabla \cdot \vec{u} \\ &= -P\delta_{IJ} + 2\mu (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ}) \end{aligned}$$

- Then,

$$\begin{aligned} \rho \frac{D\vec{u}}{Dt} &= \rho \vec{f} + \nabla \cdot \tau_{IJ} \\ &= \rho \vec{f} + \nabla \cdot (-P\delta_{IJ} + 2\mu (S_{IJ} - \frac{1}{3} \Delta \delta_{IJ})) \\ &= \rho \vec{f} - \nabla \cdot P\delta_{IJ} + \nabla \cdot 2\mu S_{IJ} - \nabla \cdot \frac{2}{3}\mu \Delta \delta_{IJ} \\ &= \rho \vec{f} - \nabla P + \nabla \cdot \mu \left(\frac{\partial u_I}{\partial x_J} + \frac{\partial u_J}{\partial x_I} \right) - \nabla \cdot \frac{2}{3}\mu \Delta \delta_{IJ} ; \text{ where } \nabla \cdot P\delta_{IJ} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} = \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \right) = \nabla P \\ &= \rho \vec{f} - \nabla P + \frac{d}{dx} \left(\mu \left(\frac{\partial u_I}{\partial x_J} + \frac{\partial u_J}{\partial x_I} \right) \right) - \frac{d}{dx} \left(\frac{2}{3}\mu \Delta \delta_{IJ} \right) \\ &= \rho \vec{f} - \nabla P + \mu \frac{\partial^2 u_I}{\partial x_J \partial x_I} + \mu \frac{\partial^2 u_J}{\partial x_I \partial x_J} - \frac{2}{3}\mu \frac{d}{dx} (\Delta \delta_{IJ}) \\ &= \rho \vec{f} - \nabla P + \mu \frac{\partial^2 u_I}{\partial x_J \partial x_I} + \mu \frac{d}{dx} \left(\frac{\partial u_I}{\partial x_J} \right) - \frac{2}{3}\mu \frac{d}{dx} \left(\frac{\partial u_I}{\partial x_J} \right) ; \text{ where } \nabla \cdot \Delta \delta_{IJ} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} = \left(\frac{\partial \Delta}{\partial x}, \frac{\partial \Delta}{\partial y}, \frac{\partial \Delta}{\partial z} \right) = \nabla \Delta \\ &= \rho \vec{f} - \nabla P + \mu (\nabla \cdot \vec{u}) + \mu \nabla \Delta - \frac{2}{3}\mu \nabla \Delta \\ &= \rho \vec{f} - \nabla P + \mu \nabla^2 \vec{u} + \frac{\mu}{3} \nabla \Delta \end{aligned}$$

$\xrightarrow{\text{From: } \frac{\partial P}{\partial x} + 0 + 0 = \frac{\partial P}{\partial x} \text{ for } x_1 \text{ position}}$

Question) Derive the Energy Equation.

Answer)

- To begin with, we need to note that the method as indicated below will be used to derive the equation;

	Lagrangian	Eulerian
Infinitesimal		: Material volume is used (V_{el}).
Control Volume	V	↳ It always consists of the same fluid elements which can travel.

- Based on the 1ST law of thermodynamics,

$$de = \delta q + \delta w$$

⋮ ⋮ ⋮

Ⓐ Ⓑ Ⓒ

- For Ⓐ,

$$\begin{aligned} \text{Total Energy} &= e + \frac{1}{2} \vec{u} \cdot \vec{u} + \rho gh^{\infty} \quad (\text{per unit mass}) \\ &= \iiint_{V_{el}} (e + \frac{1}{2} \vec{u} \cdot \vec{u}) \rho dV ; \text{ where } \rho = \frac{dm}{dV} \end{aligned}$$

Thus,

$$\frac{dE}{dt} = \iiint_{V_{el}} \rho \frac{D}{Dt} (e + \frac{1}{2} \vec{u} \cdot \vec{u}) dV$$

↳ Because we are following a moving fluid element, the time rate of change is given by Substantial derivative.

- For Ⓑ,



According to the Fourier's law (the law of heat conduction), $\vec{q} = -K \nabla T$; where K = thermal conductivity

↓ This is needed to explain the direction of thermal conduction ($T_1 \rightarrow T_2$)

i.e. 1-D shear flow $q = -K \frac{dT}{dy} = -K x - \text{Something} \left(\frac{T_2 - T_1}{y_2 - y_1} \right)$
= positive

Then, how much heat is transferred to the surface?

$$\begin{aligned} \iint_{A_{el}} \vec{q} d\vec{s} &= \iint_{A_{el}} -K \nabla T (-n) ds \quad (\because \text{heat is added into the surface, which is opposite to normal surface direction}) \\ &= \iint_{A_{el}} K \nabla T \vec{n} ds \end{aligned}$$

Using the Divergence theorem, $\iint_{A_{el}} K \nabla T \vec{n} ds = \iiint_{V_{el}} \frac{d}{dx_j} (K \nabla T) dv$

- For Ⓒ,

In terms of body force, $\iiint_{V_{el}} (\vec{f} \cdot \vec{u}) \rho dV$ ($\because \delta w$, rate of work done, is $\vec{F} \times \vec{u}$)

In terms of surface force, $\iint_{A_{el}} (Z_{ij} \cdot \vec{u}) d\vec{s} = \iiint_{V_{el}} \frac{d}{dx_j} (Z_{ij} \cdot \vec{u}) dv$

Hence, $C = \iiint_{V_{el}} (\vec{f}_i u_i) \rho dV + \iiint_{V_{el}} \frac{d}{dx_j} (Z_{ij} u_i) dv$

- Finally, let them gather together, then we have;

$$de = \delta q + \delta w$$

$$\Leftrightarrow \iiint_{V_{el}} \rho \frac{D}{Dt} (e + \frac{1}{2} \vec{u} \cdot \vec{u}) dV = \iiint_{V_{el}} \frac{d}{dx_j} (K \nabla T) dv + \iiint_{V_{el}} (\vec{f}_i u_i) \rho dV + \iiint_{V_{el}} \frac{d}{dx_j} (Z_{ij} u_i) dv$$

$$\Leftrightarrow \rho \frac{D}{Dt} \left(e + \frac{1}{2} u_i u_i \right) = \frac{\partial}{\partial x_j} \left(k \frac{\partial T}{\partial x_j} \right) + \rho f_i u_i + \frac{\partial}{\partial x_j} \left(\tau_{ij} u_i \right)$$

Question) Listing the Energy Equation, derive kinetic / internal / enthalpy energy Equation.

Answer) Refer to AE6009 hand-written notes

c) Kinetic energy Equation of Homework 7, 1 (a)

- As we know, Kinetic energy equation will look like;

$$\frac{D}{Dt} \left(\frac{1}{2} u_i u_i \right) \text{ in tensor form}$$

- By expanding out the material derivative, we have

$$\frac{D}{Dt} \left(\frac{1}{2} u_i u_i \right) = u_i \frac{Du_i}{Dt}$$

- Now, let's dive into the derivation of the equation.

$$\frac{D}{Dt} \left(\frac{1}{2} u_i u_i \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} u_i u_i \right) + u_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} u_i u_i \right)$$

$$* = u_i \frac{\partial u_i}{\partial t} + u_j u_i \frac{\partial u_i}{\partial x_j}$$

$$= u_i \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right)$$

$$* = u_i \frac{Du_i}{Dt}$$

$$* y = [f(x)]^n \rightarrow y' = n f(x)^{n-1} \cdot f'(x)$$

$$\text{Likewise, } \frac{1}{2} \frac{\partial}{\partial t} u_i^2 = \frac{1}{2} \cdot 2 u_i^{2-1} \frac{\partial u_i}{\partial t} = u_i \frac{\partial u_i}{\partial t}$$

$$* \frac{D(\cdot)}{Dt} = \frac{d(\cdot)}{dt} + \frac{1}{2} \cdot \nabla \cdot \frac{1}{2} \quad (\cdot) \cdot \frac{1}{2}$$

d) Internal energy Equation of Homework 7, 1(b)

- In order to obtain internal energy equation, the main idea of derivation will look like;

$$\text{Internal energy} = \text{Total energy} - \text{Kinetic energy}$$

$$\rho \frac{D e}{Dt} = \rho \frac{D}{Dt} \left(e + \frac{1}{2} \vec{u} \cdot \vec{u} \right) - \rho \frac{D}{Dt} \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right)$$

- As a result,

$$\rho \frac{De}{Dt} = Z_{JT} \frac{f u_i}{J X_J} + \frac{\rho}{J X_T} \left(K \frac{dT}{J X_T} \right)$$

- Now let's see how it worked out
- The derivation process would be done if we consider

- Recall momentum equation in tensor form
- Multiply by u_i  to make K.E. term
- Subtract the result equation off from total Energy Eq.

- Let's recall the momentum Equation.

$$\rho \frac{DU_i}{Dt} = \rho f_i + \frac{\partial Z_{JT}}{\partial X_J}$$

Here, Z_{JT} term didn't expand

• By multiplying U_T , we have

$$U_T \rho \frac{DU_T}{DT} = U_T \rho f_T + U_T \frac{\rho Z_{JT}}{\rho K_T}$$

• If we take the result from Kinetic Energy Equation,

$$\begin{aligned} \rho \frac{D}{DT} \left(\frac{1}{2} U_T U_I \right) &= U_T \rho f_T + U_I \frac{\rho Z_{JT}}{\rho K_T} \quad (\because \rho U_T \frac{DU_T}{DT} = \rho \frac{D}{DT} \left(\frac{1}{2} U_T U_I \right)) \\ * &= \rho U_I f_I + \frac{\rho}{\rho K_T} (U_I Z_{JT}) - Z_{JT} \frac{\rho U_I}{\rho K_T} \end{aligned}$$

$$f \frac{dg}{dx} = \frac{d}{dx} (fg) - g \frac{df}{dx}$$

• Finally,

$$e = e_{tot} - K.E.$$

$$\begin{aligned} \rho \frac{De}{DT} &= \left\{ \rho f_T U_I + \frac{\rho}{\rho K_T} (Z_{JT} U_I) + \frac{d}{dT} \left(K \frac{\partial T}{\partial K_T} \right) \right\} \\ &\quad - \left\{ \rho f_I U_I + \frac{\rho}{\rho K_T} (Z_{JT} U_I) - Z_{JT} \frac{\rho U_I}{\rho K_T} \right\} \end{aligned}$$

Hence,

$$\rho \frac{De}{DT} = \frac{d}{dT} \left(K \frac{\partial T}{\partial K_T} \right) + \underline{Z_{JT} \frac{\rho U_I}{\rho K_T}}$$

didn't expand at all

Just dummy scripts

- It's time to expand the term for Newtonian fluid.

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} - \bar{u}_{ij} &= \bar{u}_{ji} (S_{ij} + \bar{h}_{ij}) \\ &= \bar{u}_{ij} S_{ij} + \cancel{\bar{u}_{ij}}^0 \quad ; \text{ where } \bar{u}_{ij} = \bar{u}_{ji} \end{aligned}$$

Please note that we had already discussed about:

$$\frac{\partial u_i}{\partial x_j} = S_{ij} (\text{strain rate}) + R_{ij} (\text{rotation rate})$$

$$\bar{u}_{ij} = \bar{u}_{ji} \quad (\text{symmetric})$$

$$h_{ij} = -h_{ji} \quad (\text{anti-symmetric})$$

Also, we know that (for Newtonian fluid):

$$\bar{u}_{ij} = \bar{u}_{ji} = -P \delta_{ij} + 2\mu (S_{ij} - \frac{1}{3} \Delta \delta_{ij})$$

Hence, of. Homework 7, 1(c)

$$\begin{aligned} \bar{u}_{ij} \frac{\partial u_i}{\partial x_j} &= \bar{u}_{ij} S_{ij} = S_{ij} \left\{ -P \delta_{ij} + 2\mu \left(S_{ij} - \frac{1}{3} \Delta \delta_{ij} \right) \right\} \\ &= -P S_{ii} \delta_{ii} + 2\mu S_{ij} \left(S_{ij} - \frac{1}{3} \Delta \delta_{ij} \right) \\ &= -P S_{ii} + 2\mu S_{ij} \left(S_{ij} - \frac{1}{3} \Delta \delta_{ij} \right) \\ &= -P \Delta + \mu \Phi \end{aligned}$$

where $\Delta = \text{strain rate}$, $\Phi = 2 S_{ii} (S_{ii} - \frac{1}{3} \Delta \delta_{ii})$

As we substitute the result into the Internal Energy Equation,

$$\rho \frac{de}{dt} = -P\Delta + \mu \bar{\epsilon} + \frac{d}{dx_T} \left(K \frac{dT}{dx_T} \right)$$

Here are some observations ;

- 1) $\Delta < 0$ (If being compressed), $-P\Delta > 0$ (work done on fluid)
- 2) If we think about viscous effect, we expect to see $\bar{\epsilon} \gg 0$

e) Enthalpy Equation of Homework, 1 (d)

In general, the Enthalpy is defined as following ;

$$\text{Enthalpy (H)} = \text{Internal energy (e)} + \text{flow work (PV)}$$

If we consider the enthalpy per unit mass,

$$h = e + \frac{P}{\rho} \quad ; \text{ where } v = \frac{m}{\rho} \rightarrow dv = \frac{1}{\rho}$$

$$\text{Then, } dh = de + d\left(\frac{P}{\rho}\right)$$

$$= de + \frac{\rho dp - Pdv}{\rho^2}$$

In order to derive the Enthalpy Equation, let's rearrange the equation and use the concept of substantial derivative.

• By multiplying ρ ,

$$\rho \frac{dh}{dt} = \rho \frac{de}{dt} + \frac{DP}{dt} - \frac{P}{\rho} \frac{DP}{dt}$$

• If we substitute $\rho \frac{de}{dt}$ equation into the equation above,

$$\begin{aligned}\rho \frac{dh}{dt} &= -P\Delta + \mu \bar{\epsilon} + \frac{d}{dx_T} (k \frac{dT}{dx_T}) + \frac{DP}{dt} - \frac{P}{\rho} \frac{DP}{dt} \\ &= \mu \bar{\epsilon} + \nabla \cdot (k \nabla T) + \frac{DP}{dt} - P\Delta - \frac{P}{\rho} \frac{DP}{dt} \\ &= \mu \bar{\epsilon} + \nabla \cdot (k \nabla T) + \frac{DP}{dt} - P \left(\cancel{\Delta} + \frac{1}{\rho} \frac{DP}{dt} \right) \\ &\quad * \rightarrow 0 \text{ (Continuity Equation)}\end{aligned}$$

$$\therefore \rho \frac{dh}{dt} = \frac{DP}{dt} + \nabla \cdot (k \nabla T) + \mu \bar{\epsilon}$$

* From continuity equation, we have

$$\frac{DP}{dt} + \rho \nabla \cdot \vec{U} = 0 \Leftrightarrow \frac{DP}{dt} = -\rho \Delta$$

• If calorically perfect gas, $h = c_p T$; where $c_p = \text{constant}$

$$\rho c_p \frac{dT}{dt} = \frac{DP}{dt} + \nabla \cdot (k \nabla T) + \mu \bar{\epsilon}$$