

Prandtl's lifting line theory (Mock Qual)

Tuesday, December 26, 2017 02:01

For the glory of God

§. Introduction

- We studied thin airfoil theory to analyze airfoil properties.

⇒ Indeed, airfoil data are frequently denoted as infinite wing data.

- However, all real airplanes have wings of finite span.
- In fact, an airfoil is simply a section of a wing, thus, you might expect the wing to behave exactly the same as the airfoil. ⇒ However, it is not correct.
- Then, "why are the aerodynamic characteristics of a finite wing different from the properties its airfoil sections?"
 - A finite wing is a three-dimensional body; that is, there is a component of flow in the spanwise direction.
 - The flow near the wing tips tends to curl around the tips. As a result, on the top surface of the wing, there is generally a spanwise component of flow from the tip toward the wing root.

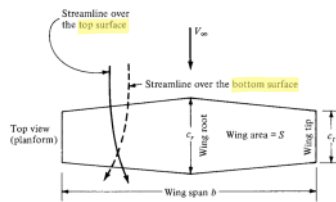
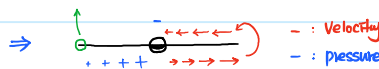


Figure 5.3 Finite wing. In this figure, the curvature of the streamlines over the top and bottom of the wing is exaggerated for clarity.

- Velocity must exist
- No pressure difference at tip



- Therefore, you would expect the overall aerodynamic properties of such a wing to differ from those of its airfoil sections.
- e.g. Maximum point of sectional lift of 3D wing is not equal to 2D airfoil because of wing tip vortices. ($C_x \neq C_L$)

§. Downwash and induced drag

- This flow establishes a circulatory motion that trails downstream of the wing; that is, a trailing vortex is created at tip.

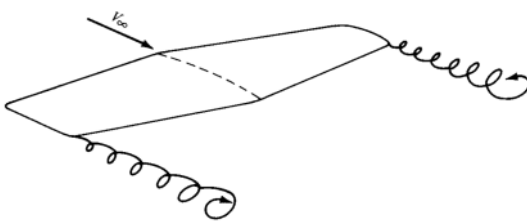


Figure 5.4 Schematic of wing-tip vortices.

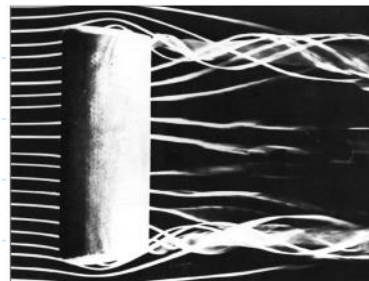


Figure 5.5 Wing-tip vortices from a rectangular wing. The wing is in a smoke tunnel, where individual streamlines are made visible by means of smoke elements. (Source: Head, M. R., in *Flow Visualization II*, W. Metzger (Ed.), Hemisphere Publishing Co., New York, 1982, pp. 399–403. Also available in Van Dyke, Milton, *An Album of Fluid Motion*, The Parabolic Press, Stanford, CA, 1982.)

- These wing-tip vortices downstream of the wing induce a small downward component of air velocity in the neighborhood of

the wing itself.

↳ This is called as downwash and denoted by w .

In turn, the downwash combines with the freestream velocity.

- Hence, the presence of downwash over a finite wing reduces the angle of attack that each section effectively sees.
- Moreover, it creates a component of drag, namely the induced drag (D_i).

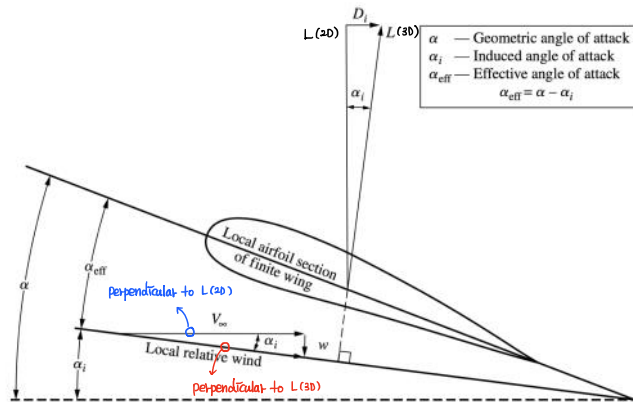


Figure 5.6 Effect of downwash on the local flow over a local airfoil section of a finite wing.

§. Additional tools needed for finite wing analysis

So, it seems like that thin airfoil theory is not valid for finite wing analysis due to the reasons mentioned above.

Then, how would you model the finite wing?

⇒ Prandtl's classical lifting line theory

Before we dive into the topic, here are some tools that we need to study first.

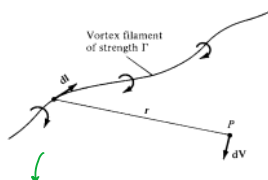
The Biot-Savart Law & Helmholtz's theorem

The law is one of the most fundamental relations in the theory of inviscid and incompressible flow.

↳ It tells us : (The Biot-Savart law)

$$d\vec{v} = \frac{\Gamma}{4\pi} \frac{d\vec{x} \times \vec{r}}{|\vec{r}|^3}$$

; where



$d\vec{x}$: Segment of the vortex filament

\vec{r} : The radius vector from $d\vec{x}$ to an arbitrary point P in space

$d\vec{v}$: Induced velocity at point P by the vortex filament segment $d\vec{x}$

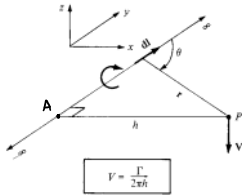
Γ : The strength of the vortex filament

In general, a vortex filament can be curved.

Helmholtz established several basic principles of vortex behavior :

- 1) The strength of a vortex filament is constant along its length.
- 2) A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid, which can be $\pm \infty$, or form a closed path.

Let me apply the law to a straight vortex filament of infinite length as below :



Here ;

Let h be the perpendicular distance from point P to the vortex filament



$\left\{ \begin{array}{l} \text{as } \theta \rightarrow 0, dl \parallel dr, \rightarrow dl \rightarrow dr (\infty) \\ \text{as } \theta \rightarrow \pi, dl \parallel dr, \rightarrow dl \leftarrow dr (-\infty) \end{array} \right.$

- The velocity induced at P by the entire vortex filament ;

$$\vec{V} = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi} \frac{d\vec{x} \times \vec{r}}{|\vec{r}|^3}$$

- The magnitude of the velocity is ;

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{dl \cdot r \cdot \sin\theta}{r^3} = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta}{r^2} dl$$

- According to the FBD, we have ;

$$r = \frac{h}{\sin\theta} \text{ and } l = \frac{h}{\tan\theta} \Rightarrow dl = h \cot\theta' = -h \csc^2\theta = -\frac{h}{\sin^2\theta} d\theta$$

- By substituting, we have

$$V = -\frac{\Gamma}{4\pi} \int_{\pi}^0 \frac{\sin\theta}{\frac{h^2}{\sin^2\theta}} \frac{h}{\sin^2\theta} d\theta = -\frac{\Gamma}{4\pi h} \int_{\pi}^0 \sin\theta d\theta = \frac{\Gamma}{2\pi h}$$

Note that (-) sign doesn't appear in the equation because we simply derived the absolute magnitude of V .

§. Prandtl's classical Lifting line theory

a) Introduction

- The first practical theory for predicting the Aerodynamic properties of a finite wing was developed by Ludwig Prandtl and his colleagues at Göttingen in Germany during the period 1911 - 1918. (World War I)
- The utility of the theory is so great that it is still in use today for preliminary calculations.
- This theory is accurate when the wing is high AR / No swept angle / Taper ratio ≈ 1 / thin airfoil.
- Also, the flow is assumed to be inviscid, incompressible, and irrotational flow.

b) Single horseshoe vortex

He postulated it is thin but I'd say that it is not necessary to be thin only.

- Prandtl reasoned as follows ;

- Bound vortex will experience a force $L' = \rho_{\infty} U_{\infty}^2 \Gamma$ from the Kutta-Joukowski theorem.
- He tried to replace a finite wing of span b with a bound vortex from $y = -b/2$ to $y = b/2$.

↳ In the meantime, due to Helmholtz's theorem, he assumed the vortex filament continues as two

free vortices trailing downstream from the wing tip to infinity.

- He ended up calling it a horseshoe vortex, which was used for modeling a finite wing, because of the shape.

Note that this is illustrated with only three horseshoe vortices for the sake of clarity

- When infinite number of horseshoe vortices are superimposed along the lifting line, we will see :
 - Vertical bars have become a continuous distribution of $\Gamma(y)$ along the lifting line.
 - The finite number of trailing vortices have become a continuous vortex sheet trailing downstream of the lifting line.
- (Note that the vortex sheet is parallel to the direction of V_∞)
- ↳ the total strength of the sheet is zero because it consists of pairs of trailing vortices of equal strength but in opposite directions.

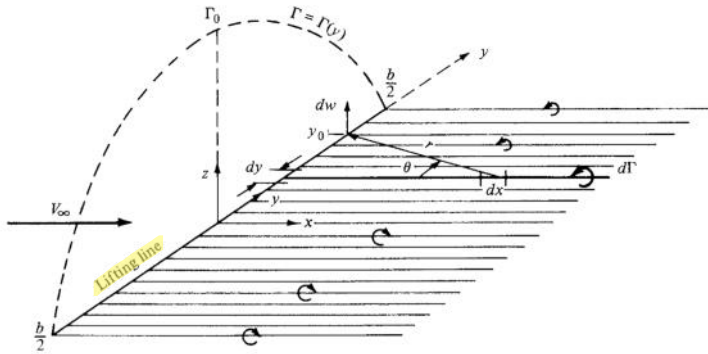


Figure 5.15 Superposition of an infinite number of horseshoe vortices along the lifting line.

Here, when we calculate Biot-Savart law,

We care about all trailing vortices but would neglect the bound vortex itself.

d) Fundamental Equation of Prandtl's Lifting Line theory.

- We have replaced the finite wing with the model of a lifting line along which the circulation $\Gamma(y)$ varies continuously.
- After all, we want to calculate $\Gamma(y)$ for a given finite wing, so that we can calculate $L' = \rho_\infty U_\infty \Gamma'$
- The fundamental equation (the only unknown is Γ') is simply derived from the fact that :
 - the geometric angle of attack is equal to the sum of the effective angle plus the induced angle of attack.

$$\alpha_{\text{geometry}} = \alpha_{\text{effective}} + \alpha_{\text{induced}}$$

③ ② ①

In terms of ①,

- Let's consider the arbitrary location y_0 along the lifting line.
- The induced angle of attack is

$$\tan \alpha_i(y_0) = \frac{w(y_0)}{U_\infty} \Leftrightarrow \alpha_i(y_0) = \tan^{-1} \left(\frac{w(y_0)}{U_\infty} \right)$$

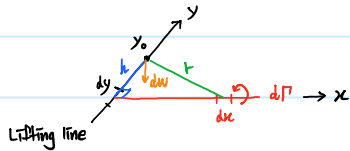
- Generally, w is much smaller than U_∞ and hence α_i is a small angle.

for small angle, $\alpha_i(y_0) \simeq \frac{w(y_0)}{U_\infty}$; where $w(y_0)$ is the value of the downwash at y_0 due to all the trailing vortices.

- In order to get the expression of $w(y_0)$, let us single out small segment of the lifting line dy located at the y coordinate.



$$dw = - \frac{(d\Gamma/dy) dy}{4\pi (z - y)} \quad ; \quad \text{where } d\Gamma = \frac{d\Gamma}{dy} dy$$



$$dw = - \frac{(d\Gamma/dy) dy}{4\pi(y_0 - y)} \quad ; \text{ where } d\Gamma = \frac{d\Gamma}{dy} dy$$

; Here, any segment of the trailing vortex dx will induce a velocity w at y_0 given by the Biot-Savart law.

$$w = \frac{\Gamma}{4\pi b} \text{ by entire semi-infinite vortex filament}$$

- Hence, the total velocity w induced at y_0 by the entire trailing vortex sheet is ;

$$w(y_0) = - \frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) dy}{y_0 - y}$$

- Substitute $w(y_0)$ into induced angle of attack equation, we have

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) dy}{y_0 - y} \quad ; \text{ an expression for the induced angle of attack in terms of the circulation distribution } \Gamma(y) \text{ along the wing}$$

- In terms of ②.

- Since the downwash varies across the span, then α_{eff} is also variable.

- According to the thin airfoil theory,

$$C_L = 2\pi \alpha = 2\pi (\alpha_{eff} - \alpha_{L=0}) \quad ; \text{ this is why, for example, } \alpha_{L=0} = -2 \quad \text{Let's say } \begin{cases} \alpha_{geo} = +2 \\ \alpha_{eff} = 0 \text{ (due to } w) \\ \alpha_{L=0} = -2 \end{cases} \therefore 1 = 0 - (-2)$$

- In any event, $\alpha_{L=0}$ is a known property of the local airfoil sections ; but
 if Aerodynamic twist, $\alpha_{L=0}$ varies with y_0
 if no Aerodynamic twist, $\alpha_{L=0}$ is constant across the span

- Based on K-J theorem and lift definition, we have

$$L' = \frac{1}{2} \rho_\infty U_\infty^2 C(y_0) C_L = \rho_\infty U_\infty \Gamma(y_0)$$

$$\Leftrightarrow C_L = \frac{2\Gamma(y_0)}{U_\infty C(y_0)}$$

- Then, recall the result from the thin airfoil theory,

$$2\pi (\alpha_{eff} - \alpha_{L=0}) = \frac{2\Gamma(y_0)}{U_\infty C(y_0)}$$

- By rearranging the equation with respect to α_{eff} , we have

$$\alpha_{eff} = \frac{\Gamma(y_0)}{\pi V_\infty C(y_0)} + \alpha_{L=0}(y_0)$$

- In terms of ③.

- Let's put them together, then we have

$$\alpha_{geometry} = \alpha_{eff} + \alpha_i$$

$$\Leftrightarrow \alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy) dy}{y_0 - y}$$

\therefore This is the fundamental Equation of Prandtl's lifting line theory. (Integro-differential Equation)

Unknown : Γ

\therefore One Equation & one variable

Known : $\alpha(y_0)$, $c(y_0)$, U_∞ , $d_{L=0}(y_0)$

\hookrightarrow A solution of Equation yields $\Gamma = \Gamma(y_0)$

of Glauert's Fourier Substitution $\Gamma(y) \rightarrow \Gamma(\beta)$

; where y_0 ranges along the span $-b/2 \sim b/2$

E) Summary of. We can't determine how the lift is distributed along the chord, as a result, the theory can't provide the pitching moment.

- Replacement of the finite wing with an infinite number of horseshoe vortices along the lifting line (since the distribution is collapsed to a single line along $1/4c$)
- The circulation distribution along the span $\Gamma(y)$ is obtained from the fundamental Equation of lifting line theory
- The lift distribution is obtained from the Kutta-Joukowski theorem. $L'(y) = \rho_\infty U_\infty \Gamma(y)$ for Inviscid / Incompressible / Irrotational
- The total lift is obtained by integrating ; $L = \int_{-b/2}^{b/2} \rho_\infty U_\infty \Gamma(y) dy$
- The lift coefficient is obtained by ; $C_L = \frac{L}{q_\infty S}$
- The induced drag is obtained by ; $D_i = L \sin \alpha_i \approx L \alpha_i$ for small angle (α_i can be obtained as long as we get Γ)
- The induced drag coefficient is obtained by ; $C_{Di} = \frac{D_i}{q_\infty S}$

Thus, a solution of the fundamental Equation of Prandtl's lifting line theory for $\Gamma(y)$ is clearly the key to obtaining Aerodynamic characteristics of a finite wing.

§. Advanced Lifting line theory

- Prandtl's classical lifting line theory is only valid for high aspect ratio wing such as $AR = 6$ or 8 .
rectangular
- What if we have a wing with swept back angle instead of the rectangular one?
($2 \sim 7^\circ$)
 \rightarrow Weissinger's L-method : it is for moderate aspect ratio and swept angle $\approx \pm 15^\circ$
- What if we have a low aspect ratio wing instead of $AR = 6$ or 8 ?
 \rightarrow There is no guarantee
- \rightarrow Jones' reduction for low aspect ratio wing \rightarrow Even if Kutta condition is not satisfied, it still gives lift.

\hookrightarrow In fact, after Jones, some people were striving to enhance the theory and finally

Slender wing theory (Delta) for subsonic was introduced.

- In this theory, Polhamus suction analogy was showing up.

\hookrightarrow It is a simple method of estimating vortex lift by estimating a suction peak.

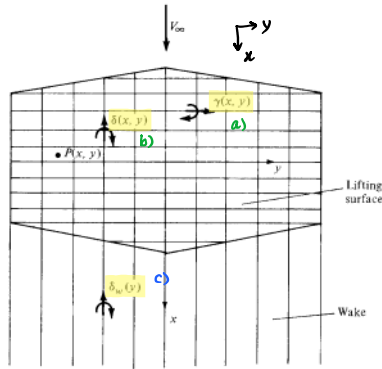
: It states that the extra normal force which is produced by a highly swept wing at high angle of attack is equal to the loss of leading edge suction associated with the separated flow.

§. Vortex Lattice Method (Analysis of finite wing)

- Prandtl's Lifting line theory gives reasonable results for straight wings at moderate to high aspect ratio.

↳ However, for some cases such as swept wing and low AR wing, the theory is inappropriate.

Let's extend the theory by placing a series of lifting lines on the plane of the wing as following :



The two vortex sheets

a) The one with vortex lines running parallel to y with strength Γ

$\Gamma = \Gamma(x, y)$; Spanwise vortex strength distribution

b) The other with vortex lines running parallel to x with strength δ

$\delta = \delta(x, y)$; Chordwise vortex strength distribution

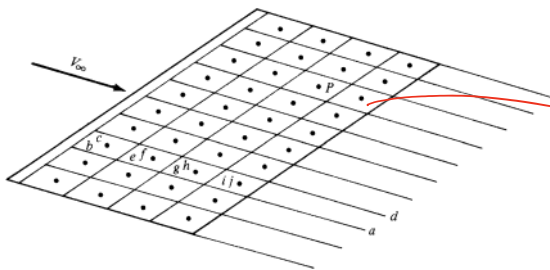
Therefore, at any given point on the surface, the strength of the lifting surface is given by both Γ and δ .

Note that c) downstream of the trailing edge

: No spanwise vortex lines but only trailing vortices.

↳ Chordwise vortices

The wing planform is divided into a number of panels as following :



; we may want the wing planform to be a stream surface of the flow

→ The entire wing is covered by this lattice of horseshoe vortices

↳ Each of different unknown strength Γ_n

→ It's said to be Von Neumann Boundary condition. (Impermeability)

⇒ can't penetrate..

- Control points on the panels can be chosen where the flow tangency condition is applied.

↳ The net normal flow velocity is zero.

- Find $\Gamma(x, y)$ and $\delta(x, y)$ such that the sum of the induced $w(x, y)$ and the normal component of the free-stream velocity to be zero for all points on the wing.

- calculate Γ and Lift

↳ the lifting surface and the wake vortex sheet induce a normal component of velocity.

↳ It can be calculated by Biot-Savart law