

# Boundary layer equations

Monday, June 19, 2017 09:32

For the glory of God

## §. Boundary layer

### a) Definition

In 1904, this concept was first introduced by Prandtl, which allowed the practical calculation of drag.

A boundary layer is the thin layer where viscous effects are dominant.

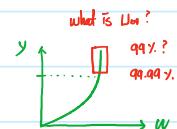
### b) BL thickness

It is the thin region of flow adjacent to a surface, where the flow is retarded by the influence of friction between a solid surface and the fluid.

The BL thickness is normally defined as the distance from the solid body to the point where

the viscous flow velocity is 99% of the free-stream velocity.

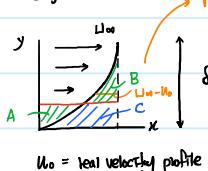
→ Displacement and momentum thickness has been introduced because the concept is quite arbitrary.



Then, what is displacement thickness?

It is an alternative concept of Boundary layer thickness, it states that the Boundary layer can be represented as a deficit in mass flow compared to Inviscid flow with slip at the wall.

e.g.

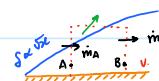


$$\text{vs. } \begin{aligned} & y \\ & \uparrow \\ & U_{\infty} \end{aligned}$$

$m = \rho U_{\infty} \delta^*$

$U_0 = U_{\infty} \quad (\text{Inviscid flow})$

of. No-slip condition is the cause of the large velocity gradients within the B.L.



Here, absolutely,  $\dot{m}_A > \dot{m}_B$  ( $\because U_A > U_B$ )

⇒ we know that mass should be conserved in the CV

∴ mass flow deficit ( $\dot{m}_A - \dot{m}_B$ ) displaced toward outside BL (↗)

↳ note  $v > 0$  (not 90 degrees)

↓

In terms of the wall direction, flow can't go

usually, negligible

Then, we know

From Inviscid flow, we had amount of C deficit in mass flow

$$D = A + C$$

$$A = B$$

$$D = B + C$$

Hence, we have

$$(or \int_0^\infty (1 - \frac{\rho U_0}{\rho U_{\infty}})^2 dy)$$

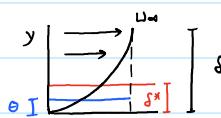
$$\rho U_{\infty} \delta^* = \rho \int_0^\infty (U_{\infty} - U_0) dy \Leftrightarrow \delta^* = \int_0^\infty (1 - \frac{U_0}{U_{\infty}}) dy \text{ if } \rho \text{ is constant}$$

what about momentum thickness?

In the same way, we will obtain an expression as following:

$$\Theta \rho U_{\infty}^2 = \rho \int_0^\infty (U_{\infty} - U_0) u dy$$

$$\therefore \Theta = \int_0^\infty \frac{U_0}{U_{\infty}} (1 - \frac{U_0}{U_{\infty}}) dy$$



$\delta$  is always greater than  $\delta^*$  and  $\Theta$ .

$$\text{Shape factor (H)} = \frac{\delta^*}{\Theta} > 1 \quad (\text{or } \delta^* > \Theta, \text{ sensitive to shape of } U/U_{\infty} \text{ vs. } y)$$

↳ In fact, there is a relationship between  $\Theta$  and  $\delta^*$ .

If no BL,  $\dot{m} = \rho \int_0^\infty U_{\infty} dy$

In reality,  $\dot{m} = \rho \int_0^\infty u dy$

∴ Deficit due to BL =  $\rho \int_0^\infty (U_{\infty} - u) dy$

Then, we have  $\rho U_{\infty} \delta^* = \rho \int_0^\infty (U_{\infty} - u) dy$

Turbulent :	1.3
Laminar :	2.5
separation :	3.4

c) Shape of BL : This is only available for either flat plate or surface with weak curvature

In general, the shape of BL can be described as  $\delta \propto \sqrt{x}$

It should be obtained from order of magnitude analysis

Let's start with N-S equations.

if. convection time  $\sim \frac{L}{U}$

$$U \rightarrow \frac{U}{\delta}$$

over this convection time,

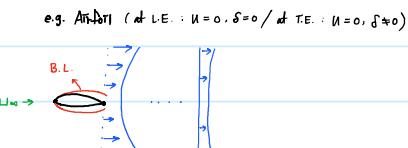
viscous effects have propagated

from wall into the fluid by a distance

$$\sim \sqrt{U L} \Rightarrow \delta \propto \sqrt{L} \Rightarrow \delta \propto \sqrt{x}$$

$$\text{Continuity equation (in vector form)} : \frac{\partial \vec{U}}{\partial t} + \nabla \cdot \vec{U} \frac{\partial \vec{U}}{\partial x} = 0$$

$$\text{Momentum equation} : \rho \frac{D \vec{U}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{U} + \frac{\mu}{3} \nabla \Delta ; \text{ where } \Delta = \nabla \cdot \vec{U}$$



Assumptions : Laminar, steady, negligible body force, constant density, two-dimensional

Momentum Equation :  $\rho \frac{D\vec{U}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{U} + \frac{\mu}{3} \nabla \Delta$ ; where  $\Delta = \nabla \cdot \vec{U}$   $\sim \sqrt{U_L U} \Rightarrow \delta \propto \sqrt{L} \approx \delta(x) \propto \sqrt{x}$

Assumptions : Laminar, steady, negligible body force, constant density, two-dimensional

1) For Continuity Equation.

if  $U$  is small but not zero  $\frac{du}{dx}$  and  $\frac{dv}{dy}$

Then, is the BL thin? so that  $\delta/L \ll 1$

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

$$O.U. \frac{U_m}{L} + \frac{V}{\delta} = 0 \Rightarrow V \sim \frac{U_m \delta}{L} \quad (\because \text{they should be in same order})$$

$$\therefore \frac{\delta}{L} \propto \frac{\sqrt{U_m L}}{L} \Leftrightarrow \sqrt{\frac{U}{L}}$$

2) For  $x$ -momentum equation,

$$\rho \left( \frac{D\vec{U}}{Dt} + \vec{U} \cdot \nabla \vec{U} \right) = \rho \vec{F} - \frac{dp}{dx} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\mu}{3} \nabla \Delta = 0$$

Steady

No body force

$$\nabla \cdot \vec{U} = 0$$

$$at A: U_A \sim U_m$$

$$at B: U_B \ll U_m \quad (\because \text{deep inside BL})$$

$$\Leftrightarrow \rho \left( U \frac{du}{dx} + V \frac{dv}{dy} \right) = - \frac{dp}{dx} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$$; \text{where } U = \frac{U_m}{\delta}$$

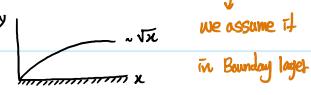
$$+ \mu \frac{\partial^2 u}{\partial y^2}$$

$$\therefore \text{Hence, } \frac{U_m^2}{L} \sim \frac{U L_m}{\delta^2} \quad (\text{if } \frac{dp}{dx} \text{ isn't significant})$$

$$\therefore \frac{U_m}{L} \sim \frac{V}{\delta}$$

$$\therefore \text{Therefore, we have } \delta \sim \sqrt{\frac{U_m L}{U_m}} ; \text{ where } L \sim x$$

↳ But, there is an argument if it is valid for turbulent, too.



we assume it  
in Boundary layer

## §. Boundary Layer Equation

↳ when turbulent, there will be a velocity fluctuation.

- Listing an order of magnitude analysis, the N-S Equations could be greatly simplified within the Boundary layer. This comes from the assumption of nearly ideal mainstream flow

with high Reynolds number such that  $\delta \sim \sqrt{\frac{UL}{U_m}} \sim \frac{L}{\sqrt{Re}}$ ; where  $\delta \ll L$  (Low Re flow, not valid)

- Let us start with Full N-S Equation

$$f. Re = \frac{UL}{\nu} = \frac{UL}{V} ; \text{ where } V = \frac{U_m}{\delta}$$

a) Continuity Equation :  $\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{U} = 0$

In other words, the essence of the BL approximation

b) Momentum Equation :  $\rho \frac{D\vec{U}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{U} + \frac{\mu}{3} \nabla \Delta$  is that the BL must be thin. (Also, not valid for separation flow)

Assumptions : Constant density flow, negligible body force, and 2D Laminar

Continuity :  $\nabla \cdot \vec{U} = 0 \Leftrightarrow \frac{du}{dx} + \frac{dv}{dy} = 0$  (In fact, continuity equation is not affected by high Re number consideration.)

Momentum | x-direction :  $\frac{du}{dt} + U \frac{du}{dx} + V \frac{dv}{dy} = - \frac{1}{\rho} \frac{dp}{dx} + V \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$

| y-direction :  $\frac{dv}{dt} + U \frac{du}{dx} + V \frac{dv}{dy} = - \frac{1}{\rho} \frac{dp}{dy} + V \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$

Listing the Order of Magnitude analysis, from now on, let us assume the flow is steady

Continuity :  $V \sim \frac{U_m \delta}{L}$  f. we already have this one

x-momentum :  $\frac{U_m}{A} + \frac{U_m^2}{L} + \left( \frac{U_m \delta}{L} \right) \frac{U_m}{\delta} = ? + V \left( \frac{U_m}{L} + \frac{U_m}{\delta^2} \right)$   
c: steady

please make sure that we talk about high Re flows, o (c:  $\delta \ll L$ ) o  $\frac{\delta}{L} = \frac{\text{small}}{\text{large}} \approx 0$

y-momentum :  $\frac{1}{A} \left( \frac{U_m S}{L} \right) + \left( \frac{U_m^2}{L} \right) \frac{U_m}{L} + \left( \frac{U_m \delta}{L} \right)^2 \frac{U_m}{\delta} = ? + V \left[ \frac{1}{L^2} \left( \frac{U_m \delta}{L} \right) + \frac{1}{\delta^2} \left( \frac{U_m \delta}{L} \right) \right]$   
o (c: steady) o (c:  $\delta \ll L$ ) o

we may have same result of the shape :

$$\frac{\delta}{L} \sim \sqrt{\frac{U}{U_m L}} \Leftrightarrow \delta \sim \sqrt{\frac{UL}{U_m}}$$

Please make sure that this is not exactly equal to zero but smaller than others

This is equal to zero because  $\frac{\partial^2 u}{\partial y^2}$  is usually dominant term in BL flows

To be more specific, in the end, after O.U.

- We end up getting the Boundary Layer Equation equipped with several assumptions!

Continuity :  $\frac{du}{dx} + \frac{dv}{dy} = 0$

If we're far from the wall,  $\frac{du}{dy} \approx 0$

\* (see below)

x-momentum :  $U \frac{du}{dx} + V \frac{dv}{dy} = - \frac{1}{\rho} \frac{dp}{dx} + V \frac{\partial^2 u}{\partial y^2}$

\* We may be able to neglect viscous term

at the point

y-momentum :  $- \frac{1}{\rho} \frac{dp}{dy} = 0$

↳ In BL approximation,  $p = p(x)$  only

$$\begin{cases} p_0 = p_0 \\ p \neq p_0 \end{cases}$$

x-momentum :  $\frac{U_m^2}{L} \sim \frac{U L_m}{\delta^2}$

smaller than corresponding term in

x-momentum by factor  $\delta/L (\ll 1)$

$$x\text{-momentum} : u \frac{du}{dx} + v \frac{dv}{dy} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad \therefore \text{We may be able to neglect viscous term}$$

$$y\text{-momentum} : - \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \quad \hookrightarrow \text{In BL approximation, } p = p(x) \text{ only}$$

$p_0 = p_\infty$   
 $p_0 \neq p_\infty$

§. BL Equation Transformation  $\frac{dp}{dy} = 0$  ( $\because \rho = \text{const}$ ) ; It can be also shown experimentally.

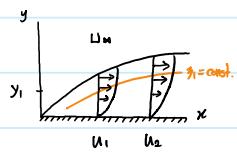
- Although the N-S Equations were greatly simplified to BL equations, the Equations are still Partial Differential Equation (P.D.E.).

→ In other words, it is hard to solve the Equations theoretically unless the Equations become Ordinary Differential Equation (O.D.E.)

- A similarity variable enables us to transform P.D.E. to O.D.E. by reducing the number of unknowns.

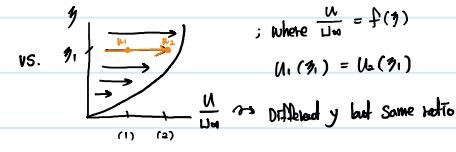
Then, what is similarity variable? The idea: Velocity profile at different location or times are similar.

→ ... ; Momentum diffusion due to viscous effects is expected more as time



is increasing. (viscous length scale,  $\sqrt{U\tau}$ )

; where  $U = f(x, y)$   
 $U_1(x_1, y_1) \approx U_2(x_2, y_2)$



\* Anderson - Hypersonic and High temperature

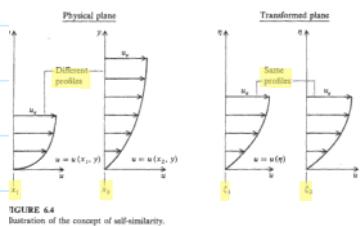


FIGURE 6.4 Illustration of the concept of self-similarity.

Hence,  $\gamma$  can be defined as; Stokes suggested that  $\gamma$  is better to measure close to the wall.

$$\gamma = \frac{\text{distance from plate}}{\text{viscous length scale}} = \frac{y}{2\sqrt{U\tau}} ; \text{Dimensionless} \quad (\tau = x/U_\infty)$$

$$\begin{cases} y: [L] \\ t: [T] \\ U: [L^2 T^{-1}] \end{cases}$$

$\gamma$  (Dimensionless)  $\equiv \frac{y}{\text{Something}}$  ; That would be better measure  $\downarrow$   
↓ : Is it close to the wall?  
↓ might be vague

Let us transform the BL Equations

From now on, let's transform the equation.

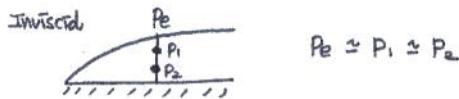
: Let's start from the Boundary Layer Equations. (Same assumptions)

$$\text{Continuity} : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$x\text{-momentum} : u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial^2 u}{\partial y^2} \dots \textcircled{1}$$

$$y\text{-momentum} : \frac{\partial p}{\partial y} \approx 0$$

- In terms of  $x$ -momentum, since  $\partial p / \partial y = 0$ , we might be able to use Bernoulli's equation because;



By recalling the Bernoulli's equation,

$$p + \frac{1}{2} \rho u^2 = \text{const} \quad \begin{matrix} \text{maybe *} \\ u = \text{general expression} \end{matrix}$$

so that  $u = u(x)$

If we assume the  $u$  is ~~constant~~<sup>\*</sup>, we have; at this point

↳ It's assumed as a constant for Blasius case

$$\frac{\partial p}{\partial x} + \rho u \frac{du}{dx} = 0 \dots \textcircled{2}$$

Substitute  $\textcircled{2}$  into  $\textcircled{1}$ , we have;

outside of BL, Bernoulli's equation,

$$\frac{dp_0}{dx} + \rho u \frac{du}{dx} = 0$$

For  $y$ -momentum,  $\frac{dp}{dy} = 0 \quad \therefore p \neq p(y), p = p_\infty$

of course,  $p_1 > p_2$  in channel flow

It means  $p_1 = \text{const.}$  as well (independent of  $x$ )

$\therefore p_1 = p_2$  on flat plate.

→ what if  $p_1 > p_2$  on flat plate?  $\uparrow$  with  $x$

(accelerating BL)

$\left. \begin{array}{l} u = \text{const} : \text{Blasius} \\ u = x^{\frac{1}{2}} : \text{Falkner-Skan} \end{array} \right\} \rightarrow$  Both are similarity solutions

we will look into it.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u \frac{du}{dx} + v \frac{\partial u}{\partial y^2} \dots x\text{-momentum}$$

Step 1 : Introduce streamfunction ( $\psi$ ) a kind of strategies

As usual, we work in terms of the streamfunction so that

the continuity equation is automatically satisfied.

$$\therefore \frac{\partial \psi}{\partial x} \quad \text{and} \quad \therefore = - \frac{\partial \psi}{\partial y}$$

the continuity equation is automatically satisfied.

$$u = \frac{\partial x}{\partial y} \quad \text{and} \quad v = -\frac{\partial x}{\partial x}$$

Then, the  $x$ -momentum equation is as following :

$$\frac{\partial u}{\partial y} \frac{\partial^2 x}{\partial x \partial y} - \frac{\partial x}{\partial x} \frac{\partial^2 x}{\partial y^2} = u \frac{\partial u}{\partial x} + v \frac{\partial^3 x}{\partial y^3}$$

Now, we have 3<sup>RD</sup> order non-linear P.D.E. but 1 unknown

Step 2 : Transformation to make P.D.E  $\rightarrow$  O.D.E

Let  $x = g(x) f(y)$ ;  $f(y)$  will be non-dimensional

We may be able to obtain :

$$\text{Let } y = \frac{uy}{g(x)}$$

$$\text{Let } \frac{x(x,y)}{y} = f(y)$$

so, where does it come from?

$$\Leftrightarrow x(x,y) = g(x) f(y); \text{ w.r.t. } f(y), \text{ we are talking about BL}$$

$\hookrightarrow$  This is because BL is a function of  $x$  as well. (Note that it has the same dimension of  $x$ )

Then, how should  $g(x)$  be chosen (depending on  $u(x)$ ) to get similarity solution?

$\Rightarrow$  Let's figure out.

Furthermore,

$f(y)$  would eventually give you  
in terms of BL such as

Since  $x = [\text{Velocity}] [\text{Length}] = L^2 T^{-1}$ , Blasts BL solutions.  
 $\therefore$  we have  $g(x)$

then  $g(x)$  should be matched to get  $y$  dimensionless one.

For this reason,

$$(\because y = \frac{y}{\text{length}}) \quad y = \frac{y}{g(x)/u} = \frac{L}{L^2 T^{-1}} = \frac{L}{LT^{-1}} = \text{Dimensionless}$$

$$\therefore \text{Thus, we have } y = \frac{uy}{g(x)}$$

To go further, let's use the chain rule. ( $x = g(x) f(y)$ )

$$\begin{aligned} \frac{\partial x}{\partial y} &= g(x) \frac{df(y)}{dy} \\ &= g(x) \frac{df(y)}{dy} \frac{dy}{dy} \end{aligned}$$

$$\therefore \text{where } \frac{\partial y}{\partial y} = \frac{u}{u}$$

$$\left. \begin{array}{l} u = \frac{\partial x}{\partial y} \\ v = -\frac{\partial x}{\partial x} \end{array} \right\}$$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$\text{; where } \frac{\partial y}{\partial y} = \frac{u}{g(x)}$$

$$\frac{du}{dx} = \frac{d g(x)}{dx} f(z) + g(x) \frac{df(z)}{dz}$$

$$= \frac{d g(x)}{dx} f(z) + g(x) \frac{df(z)}{dz} \frac{\partial z}{\partial x}$$

$$\text{; where } \frac{\partial z}{\partial x} = \frac{\frac{d u}{dx} g(x) - u \frac{d g(x)}{dx}}{g(x)^2} = \frac{d u}{dx} \frac{y}{g(x)} - \frac{u y}{g(x)^2} \frac{d g(x)}{dx}$$

Now, substitute what we did into original x-mom. Equation:

$$f''' + \frac{g(x)}{U} \frac{dg(x)}{dx} f \cdot f'' + \frac{g(x)^2}{U^2} \frac{dU}{dx} (1 - f'^2) = 0$$

For a similarity solution to exist,  $\alpha$  and  $\beta$  must be constant  
(for the P.D.E.)

$$\frac{g(x)}{U} \frac{dg(x)}{dx} = \alpha = \text{const.}$$

or,  $f' = \frac{df(\beta)}{d\beta}$

$$\frac{g(x)^2}{U^2} \frac{dU}{dx} = \beta = \text{const.}$$

If then, we changed P.D.E. to O.D.E. with only one variable  $f$ .

### e) Falkner-Skan Flows

If we think about the  $\alpha$  term, we end up realizing that we are free to choose  $\alpha = 1$ . (why? It's gonna be easy)

Since  $\frac{dg^2}{dx} = 2g \frac{dg}{dx}$ ,  $\frac{dg}{dx} = \frac{1}{2g} \frac{dg^2}{dx}$       2, 3, 4, ...  
whatever it will

$$\frac{g(x)}{U} \frac{\frac{dg(x)}{dx}}{\frac{dg^2}{dx}} = \text{const} = \alpha = 1$$

be fine  
 $(\because \text{define } m)$

$$\Leftrightarrow \frac{1}{2U} \frac{dg(x)^2}{dx} = 1 \quad (\text{If in detail, find out reference})$$

Finally, we have :

$$\frac{g}{U} \frac{dg}{dx} = 1 \quad \text{and} \quad \frac{g^2}{U U^2} \frac{dU}{dx} = \beta \quad \dots \textcircled{1}$$

In order to rearrange the **second term**,

$$\begin{aligned} \frac{d}{dx} \left( \frac{g^2}{U} \right) &= \frac{1}{U} \frac{dg^2}{dx} - \frac{g^2}{U^2} \frac{dU}{dx} \quad \dots \text{Just Maths} \\ &= 2U - \beta U \\ &= (2-\beta)U \quad \dots \text{constant} \end{aligned}$$

Finally, we have  $g^2$  term.

By Integrating,

$$g^2 = (2-\beta)U U x \quad \dots \textcircled{2}$$

Substitute  $\textcircled{2}$  into  $\textcircled{1}$ , we have :

$$\begin{aligned} \frac{g^2}{U U^2} \frac{dU}{dx} &= \beta \\ \Leftrightarrow \frac{(2-\beta)U U x}{U U^2} \frac{dU}{dx} &= \beta \\ \Leftrightarrow \frac{1}{U} \frac{dU}{dx} &= \left( \frac{\beta}{2-\beta} \right) \frac{1}{x} \quad ; \text{where } m = \frac{\beta}{2-\beta} \end{aligned}$$

.....

• By integrating,

$$\ln U = m \ln x + C_1$$

$$\Leftrightarrow \ln U = \ln x^m + C_1$$

$$\Leftrightarrow e^{\ln U} = e^{(\ln x^m + C_1)}$$

$$\Leftrightarrow U = e^{C_1} \cdot x^m \quad \therefore U(x) = C \cdot x^m$$

; This is called as Falkner-Skan Flows family.

• There are two special cases in Falkner-Skan flows.

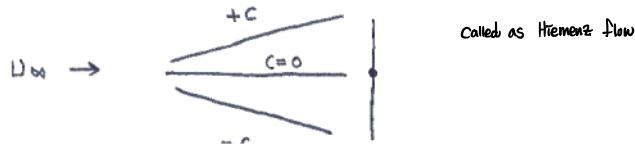
①  $m=0$  : Blasius flow over a flat plate

Since we started  
from  $P_\infty$

$$\Rightarrow U(x) = \text{constant} = U_\infty ; \text{ where } U(x) = \text{Wedge}$$



②  $m=1$  : 2D stagnation flow,  $U(x) = \text{constant} \times x$



### g) The Blasius solution for flat plate flow $\textcircled{A}$

- As we discussed it before, we have the Blasius solution when;

$$f''' + \alpha f f'' + \beta(1-f^2) = 0$$

; where  $\alpha = 1$ ,  $\beta = \text{const.}$

when  $m = 0$  ( $m = \frac{\beta}{2-\beta}$ ) which means  $\beta = 0$ , we have

$$f''' + f f'' = 0 ; \text{ Blasius Equation}$$

- From now on, let's see it in detail.

- In 1908, H. Blasius who was Ludwig Prandtl's first student found a celebrated solution for laminar boundary layer flow past a flat plate.

- As we discussed, this was also done by transforming the two partial differential equations into a single ordinary differential equation by introducing a new independent variable which is called the similarity variable.

of. For Blasius solution, the equation is exact; but the solution is not.

- He assumed that  $U = U_\infty = \text{constant}$ , which means

$$\text{Continuity} : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$x\text{-momentum} : U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U \frac{\partial U}{\partial x} + V \frac{\partial^2 U}{\partial y^2} \quad \begin{matrix} \text{this is concluded} \\ \text{by } y\text{-momentum in} \\ \text{2D flow} \end{matrix}$$

$$y\text{-momentum} : \frac{\partial p}{\partial y} \approx 0 \quad \begin{matrix} \text{: This means } \frac{\partial p}{\partial x} = 0 \\ \text{Since } U = \text{const}, \frac{\partial}{\partial x}(\text{const}) = 0 \end{matrix}$$

- Therefore, for flow over a flat plate, the pressure remains constant over the entire plate for both inside and outside.

- Boundary conditions are :

- No slip condition :  $U = V = 0$  at  $y = 0$

- Infinity condition :  $U = U_\infty$  at  $y \rightarrow \infty$

- So, it feels like that we have both governing equations and boundary conditions to solve the Blasius problem.

- Let's follow what he had concerned.

→ Next page

\* B.C.s : No-slip condition :  $u=v=0$  at  $y=0$

Infinity condition :  $u=u_e$  at  $y \rightarrow \infty$

) ... ②

Let's solve !! ( Blasius 가 고민했던 것들 살펴봅시다 )

1<sup>st</sup> step : Reduction of Number of Unknowns & eqs.

Introduce stream function :  $\psi \rightarrow$  Continuity is automatically

Def.  $\Rightarrow u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$  satisfied.

$$\therefore \frac{d}{dx} \left( \frac{\partial \psi}{\partial y} \right) - \frac{d}{dy} \left( \frac{\partial \psi}{\partial x} \right) = 0$$

Hence, Eq. of motion becomes, It was :  $\begin{cases} x\text{-momentum} : u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \\ y\text{-momentum} : \frac{\partial p}{\partial y} = 0 \end{cases}$

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} \quad \dots \quad ③$$

: 3<sup>RD</sup> order, Non-linear P.D.E for 1 Eq., 1 unknown.

2<sup>nd</sup> step : Finding transformation (PDE  $\rightarrow$  ODE)

Introduce  $\eta(x,y) \sim \frac{y}{\sqrt{x}}$  : 이런 variable 도입

[이터]

Let's define,  $\eta = \frac{y}{x} \left( \frac{u_e}{\nu x} \right)^{\frac{1}{2}}$

$$\psi = (\nu u_e x)^{\frac{1}{2}} \eta + f(y)$$

↳ Blasius realized that the shape of the velocity profile remains the same along the plate. Also, he reasoned the non-dimensional velocity profile  $u/\nu x$  should remain unchanged when plotted against the non-dimensional distance  $y/x$ .

→ Finally, he introduced the similarity parameter

where  $\gamma = \text{dimensionless}$ .

Trial & error로 해보니, ④ 일때 P.D.E  $\rightarrow$  O.D.E로 바뀌더라 ~..

Differential : chain Rule.

$$u = \frac{du}{dy} = \frac{du}{dx} \frac{dx}{dy} \xrightarrow{?} = (\nu \frac{de}{dx})^{\frac{1}{2}} f \cdot \frac{1}{2} \left( \frac{ue}{\nu x} \right)^{\frac{1}{2}} = \frac{1}{2} ue f'$$

$$\frac{du}{dx} = \frac{d}{dx} \left( \frac{du}{dy} \right) = \frac{d}{dy} \left( \frac{du}{dx} \frac{dx}{dy} \right) \frac{dy}{dx} = -\frac{1}{4} \frac{ue}{x} y f''$$

$$v = -\frac{du}{dx} = \frac{1}{2} \left( \frac{ue}{x} \right)^{\frac{1}{2}} (y f' - f)$$

$$\frac{d^2u}{dy^2} = \frac{ue}{8} \left( \frac{ue}{\nu x} \right) f'''$$

$$\frac{du}{dy} = \frac{ue}{4} \left( \frac{ue}{\nu x} \right)^{\frac{1}{2}} f''$$

직접 계산해보면..

( 유도해보면, 미분공식 이용 )

→ Prof. encouraged to student.

이런 것들을..  $u \left( \frac{du}{dx} \right) + v \frac{du}{dy} = \nu \frac{d^2u}{dy^2}$  봄기시에 대입해보면..

finally,  $\frac{1}{8} \frac{ue^2}{x} (f''' + ff'') = 0$

$$\therefore f''' + ff'' = 0 \Rightarrow P.D.E \rightarrow O.D.E \text{ satisfied!!}$$

Hence, we're solving  $f''' + ff'' = 0$ , O.D.E - Nonlinear.

3 RD step : Transform B.C.s

$$( \begin{array}{l} u = v = 0 \text{ at } y = 0, f = f' = 0 \text{ at } y = 0 \\ u = ue \text{ at } y \rightarrow \infty, f' = 2 \text{ at } y = \infty \end{array} )$$

2D stagnation flow section with  
discuss about it in detail.  
where does it come from?

4 TH step : Power Series. Let's solve !!

이정하기 전에..

지금까지의 story 정리 한번 갑시다.

of. It can be solved numerically by means of standard techniques such as the Runge-Kutta method.

$u = \sqrt{\nu \frac{ex}{x}} f(\beta) \cdot \beta = \frac{x}{2} \sqrt{\frac{ue}{\nu x}}$

$$u = \frac{du}{dy} = \sqrt{\nu \frac{ex}{x}} \frac{df}{d\beta} \frac{d\beta}{dy} = \sqrt{\nu \frac{ex}{x}} f'(\beta) \frac{1}{2} \sqrt{\frac{ue}{\nu x}}$$

Then,  $u=0$  at  $y=0$  means:

$$0 = \sqrt{\nu \frac{ex}{x}} f'(0) \frac{1}{2} \sqrt{\frac{ue}{\nu x}}$$

$$\hookrightarrow f'(0) = 0 \text{ at } \beta=0$$

\* Topic : Laminar B.L. along a flat plate

(2-D, steady, incompressible flow)

$$\text{Eq. of motion} : u \frac{du}{dx} + v \frac{dv}{dy} = \nu \frac{d^2u}{dy^2}$$

$$\text{continuity} : \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

B.C. ①  $u = v = 0$  at  $y = 0$

②  $u = U_e$  at  $y \rightarrow \infty$

However, still Non-linear P.D.E.

To solve this problem, we proceed several steps.

• 1<sup>st</sup> step : Reduction of Eqs. of unknowns

→ Introduce stream function ( $\psi$ )

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Then, Eq. of motion becomes,

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}$$

• 2<sup>nd</sup> step : P.D.E → O.D.E

Looking for variable transformation. ( $y = \gamma(x, y)$ )

Blasius insists that  $y \sim \frac{y}{\sqrt{x}}$   $\left\{ \begin{array}{l} \gamma = \frac{y}{x} \left( \frac{U_e}{\nu x} \right) \\ \alpha = (\nu U_e x)^{\frac{1}{2}} f(\gamma) \end{array} \right.$

• 3<sup>rd</sup> step : Differentiation → chain rule

Specific Equations can be referred to previous page.

Still Non-linear P.D.E, not applying B.C.

4<sup>TH</sup> step : Transform of B.C.s

$$\textcircled{1} \quad f = f' = 0 \quad \text{at} \quad y = 0 \quad (\text{No-slip condition})$$

$$\textcircled{2} \quad f' = 2 \quad \text{at} \quad y \rightarrow \infty \quad (\text{infinity condition})$$

Today, we'll discuss about ...

5<sup>TH</sup> step : Solve O.D.E with transformed B.C.s  $\rightarrow$  Power series.

Assume power series solution,

$$f = A_0 + A_1 y + \frac{A_2}{2!} y^2 + \frac{A_3}{3!} y^3 + \dots$$

from No-slip condition :  $A_0 = A_1 = 0$  For no-slip condition,  $y=0$

$$\therefore f = \frac{A_2}{2!} y^2 + \frac{A_3}{3!} y^3 + \frac{A_4}{4!} y^4 + \dots \quad f(0) = A_0 + 0 + 0 + \dots = 0 \\ \therefore A_0 = 0$$

$$f' = A_2 y + \frac{A_3}{2!} y^2 + \frac{A_4}{3!} y^3 + \dots \quad f'(0) = A_1 + 0 + 0 + \dots = 0 \\ \therefore A_1 = 0$$

$$f'' = \dots$$

$$f''' = \dots$$

Then, substitute above Equations ( $f'$ ,  $f''$ ,  $f'''$ ...) to  $\underline{f''' + ff'' = 0}$

6<sup>TH</sup> step : Substitute  $f'$ ,  $f''$ ,  $f'''$ ,  $f$   $\rightarrow$  O.D.E  $\downarrow$   
It will be solved numerically.

$$A_3 + A_4 y + \left( \frac{A_2^2}{2!} + \frac{A_3}{2!} \right) y^2 + \dots = 0$$

Here, 0이 되려면  $A_3 = A_4 = 0$ ,  $A_2^2 + A_3 = 0$ ,  $\dots = 0$

Then, 무수한 연립방정식이 나오고 임의의  $A_2$ 에 대해서 풀수있겠지.

$$\text{finally, } f = A_2^{1/3} \left[ \frac{(A_2^{1/3} y)^2}{2!} - \frac{(A_2^{1/3} y)^5}{5!} + \frac{(A_2^{1/3} y)^8}{8!} \dots \right]$$

$\hookrightarrow A_2$ 에 대해서 풀거지..

; where  $\Gamma \equiv A_2^{1/3} \gamma$

$\therefore f = A_2^{1/3} g(\Gamma)$ , or the infinity condition을 만족시키는  $A_2$ 를 찾으면 되겠다.

• 7<sup>TH</sup> step : determine the value of  $A_2$

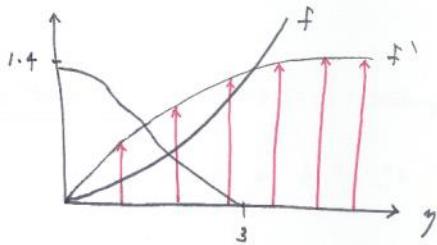
• B.C.  $\lim_{\gamma \rightarrow \infty} f' = 2$  (condition)

$$\leftarrow \lim_{\gamma \rightarrow \infty} A_2^{1/3} g'(\Gamma) \frac{d\Gamma}{d\gamma} = A_2^{2/3} g'(\Gamma)$$

$$\therefore \boxed{\lim_{\gamma \rightarrow \infty} A_2^{2/3} g'(\gamma) = 2}$$

$$\leftarrow A_2 = \left[ \lim_{\gamma \rightarrow \infty} g(\gamma) \right]^{3/2} \quad \text{where } g(\gamma) = \frac{\gamma^2}{2!} - \frac{35}{5!} + \frac{...}{8!} \dots$$

In 1938, Goldstein found  $A_2 = 1.32824 \dots$  (using several Calculus..)



"Similar solution"

$$u = \frac{1}{2} u_0 e^{\Gamma}$$

: 어떤  $\gamma$ 에 대해서  $\uparrow\uparrow$  항상 이전과 같아야 한다.. Similar solution.

그럼 이걸 가지고 무엇을 어떻게 할까?

→ Next page.

• Here, please keep in mind that :

$$f = f(\beta), \quad f' = \frac{df(\beta)}{d\beta} \approx \frac{U}{U_\infty}, \quad \text{and} \quad f'' \approx \frac{dU}{dy}$$

$\hookrightarrow$  will be maximum

④ Solve the O.D.E.

at  $y=0$

: Please see AAI binder for one example

⑤ Results summary

: All results were summarized table 4-1 in white book.

$$S_{99} = 5 \sqrt{\frac{Ux}{U_\infty}} = \frac{5x}{\sqrt{Re}} ; \quad x = \text{Reference length}$$

$$S^* = 1.7208 \sqrt{\frac{Ux}{U_\infty}} \quad \text{Derivation process is referred to Anderson ch. 18.}$$

$$\theta = 0.664 \sqrt{\frac{Ux}{U_\infty}}$$

$$C(x) = 0.664 Re_x^{-\frac{1}{2}}$$

• Here, we might be able to estimate B.L. thickness from the table summarized by Blasius.

$$y = y \sqrt{\frac{U_\infty}{2Cx}} : y \text{ tabulated and } U_\infty, U, x \text{ given} \Rightarrow y \text{ (we can get)}$$

\* Momentum Eq. "outside BL" is essentially in Bernoulli's equation.

$$\frac{P}{\rho} + \frac{1}{2} V^2 = \text{const.} ; \quad \text{where } |V| = |\vec{V}| = \sqrt{u^2 + v^2}$$

$$\text{Free-stream } U \rightarrow U_\infty, \quad V/U = 0$$

Differentiate in  $x$ ,

$$\frac{dp}{dx} + \rho U_\infty \frac{dU_\infty}{dx} = 0 \quad \left\{ \begin{array}{l} \frac{dp}{dx} = 0 \quad \text{if } U_\infty \text{ is independent of } x \\ \frac{dp}{dx} > 0 \quad \text{if } U_\infty \text{ is decreased to w.r.t. } x \end{array} \right.$$

• Since we introduced the Bernoulli's equation, flow outside BL is inviscid and irrotational.  $\frac{du}{dy} \rightarrow 0$  as  $y \rightarrow \infty$

$\hookrightarrow$  It is the case of "throughout the flow" (usually, irrotational flow at outside of BL)