Full potential equation (Mock Qual)

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For the glory of God

This question was given to me from williams Robert. (This was actually his qual question)

Question) write down full potential Equation (Continuity, momentum, energy equations are absorbed in one equation)

Answer)

$$\left[1 - \frac{1}{a^2}\left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[1 - \frac{1}{a^2}\left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - \frac{2}{a^2}\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

; where $a^2 = a_0^2 - \frac{\gamma - 1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]$

Question) What are assumptions in the Equation?

Answer)

- In general, the flow is assumed to be Inviscid, Irrotational, and compressible. (+ Isentropic flow)
- For a simplicity of math, the equation described above is assumed with steady and two-dimensional.
- When the flow is assumed to incompressible, Laplace Equation is introduced which is a linear P.D.E.
- Since the full potential Equation is a non-linear Partial Differential Equation, we need to linearize it to solve it analytically.
- Due to the assumptions made in the linearization, it is only valid if ;
  - Small perturbation, that is, thin bodies at small angles of attack
  - It is valid for both subsonic and supersonic when the equation is linearized.
- Note that this linearization is not valid for thick bodies, large α, transonic flow, and hypersonic flow. i.e. $u = U_\infty + \hat{u}$

Question) What is the final formulation of the linearization for both subsonic and supersonic?

Answer)

- For subsonic,

$$(1 - M_\infty^2)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0 \quad ; \text{ where } \hat{\phi} = \text{Perturbation velocity potential}$$

- For supersonic,

$$(M_\infty^2 - 1)\frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0$$

Question) Could you explain briefly how to derive and to linearize the equation?

Answer) Let's derive Full potential Equation first.

· Continuity Equation

$$\cancel{\frac{\partial \rho}{\partial t}} + \nabla\cdot\rho\vec{u} = 0 \iff \nabla\cdot\rho\vec{u} = 0$$

0 (∵ Steady)

$$\iff \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \quad ; \text{ for 2D flow}$$

$$\iff \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} = 0$$

Since the flow is irrotational, $\iff \rho\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial\phi}{\partial x}\dfrac{\partial\rho}{\partial x} + \rho\dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial\phi}{\partial y}\dfrac{\partial\rho}{\partial y} = 0$; where $u = \dfrac{\partial\phi}{\partial x}$ and $v = \dfrac{\partial\phi}{\partial y}$

$$\therefore \text{Continuity equation w.r.t. } \phi = \rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} = 0$$

· Momentum Equation

We are attempting to obtain an equation completely in terms of ϕ ; hence, we need to eliminate ρ from continuity eq.

0 (∵ Inviscid)

$$\rho\frac{D\vec{u}}{Dt} = \cancel{\rho\vec{f}} - \nabla p + \cancel{\mu}\nabla^2\vec{u} + \cancel{\frac{\mu}{3}}\nabla\Delta \quad ; \text{ where } \Delta = \nabla\cdot\vec{u} \text{ and } \mu = \text{const.}$$

0 (∵ Negligible body force)

$$\iff \rho\left(\cancel{\frac{\partial\vec{u}}{\partial t}} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p$$

0 (∵ Steady)

$$\iff \rho\vec{u}\cdot\nabla\vec{u} = -\nabla p$$

For x-momentum : $\rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x}$

$$\iff \rho\left(u\frac{\partial u}{\partial x} + u\frac{dy}{dx}\frac{\partial u}{\partial y}\right)dx = -\frac{\partial p}{\partial x}dx \quad ; \text{ where } \frac{u}{dx} = \frac{v}{dy} \text{ ; streamline equation}$$

$$\iff \rho u\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) = -dp$$

$$\iff \rho u\,du = -dp$$

$$\therefore dp = -\rho u\,du$$

Hence, we could say that ;

$$dp = -\rho\vec{u}\,d\vec{u} = -\rho\,d\left(\tfrac{1}{2}\vec{u}^2\right) = -\frac{\rho}{2}d(u^2+v^2) = -\frac{\rho}{2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}$$

Also, we know that ;

$$a^2 = \left.\frac{\partial p}{\partial\rho}\right|_s$$

$$\iff dp = a^2 d\rho$$

⇒ Then, $a^2 d\rho = -\dfrac{\rho}{2}d\left\{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2\right\}$

$$\iff d\rho = -\frac{\rho}{2a^2}d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}$$

$$\begin{cases} x\text{-direction} : \dfrac{\partial\rho}{\partial x}\cancel{dx} = -\dfrac{\rho}{2a^2}\dfrac{\partial}{\partial x}\cancel{dx}\left\{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2\right\} \\ y\text{-direction} : \dfrac{\partial\rho}{\partial y}\cancel{dy} = -\dfrac{\rho}{2a^2}\dfrac{\partial}{\partial y}\cancel{dy}\left\{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2\right\} \end{cases}$$

Now, recall the continuity equation ;

$$\rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} = 0$$

↳ These should be substituted to make the equation in terms of completely ϕ

$$\rho\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y}=0$$

↳ These should be substituted to make the Equation in terms of completely ϕ

; where

$$\frac{\partial\rho}{\partial x}=-\frac{\rho}{2a^2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]=-\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right)$$

cf. $\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)^2=\frac{1}{2}\times 2\frac{\partial\phi}{\partial x}\times\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)$

$$\frac{\partial\rho}{\partial y}=-\frac{\rho}{2a^2}\frac{\partial}{\partial y}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]=-\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)$$

Hence,

$$\cancel{\rho}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial\phi}{\partial x}\left\{-\frac{\cancel{\rho}}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right)\right\}+\frac{\partial\phi}{\partial y}\left\{-\frac{\cancel{\rho}}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)\right\}=0$$

$$\Leftrightarrow \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2}-\frac{1}{a^2}\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}-\frac{1}{a^2}\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2}=0$$

$$\Leftrightarrow \frac{\partial^2\phi}{\partial x^2}\left(1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right)+\frac{\partial^2\phi}{\partial y^2}\left(1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right)-\frac{2}{a^2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}=0$$

Finally,

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0$$

; It is almost completely in terms of ϕ except for a^2

· Energy Equation

a^2 can be expressed to ϕ by using special form of energy equation.

- Assumptions : Steady, Inviscid, Adiabatic, Calorically perfect gas

$$h_0=\text{const along a streamline}$$

$$\Leftrightarrow h_1+\frac{u_1^2}{2}=h_2+\frac{u_2^2}{2}$$

$$\Leftrightarrow c_pT_1+\frac{u_1^2}{2}=c_pT_2+\frac{u_2^2}{2} \quad ;\ \text{where } c_p=\frac{\gamma}{\gamma-1}R$$

$$\Leftrightarrow \frac{\gamma RT_1}{\gamma-1}+\frac{u_1^2}{2}=\frac{\gamma RT_2}{\gamma-1}+\frac{u_2^2}{2}$$

$$\Leftrightarrow \frac{a_1^2}{\gamma-1}+\frac{u_1^2}{2}=\frac{a_2^2}{\gamma-1}+\frac{u_2^2}{2} \quad ;\ \text{where } a=\sqrt{\gamma RT}$$

If we consider point 2 to be a stagnation point, where the stagnation speed of sound is denoted by a_0, then $u_2=0$

$$\therefore \frac{a^2}{\gamma-1}+\frac{u^2}{2}=\frac{a_0^2}{\gamma-1}$$

$$\Leftrightarrow a^2=a_0^2-\frac{\gamma-1}{2}u^2$$

$$=a_0^2-\frac{\gamma-1}{2}(u^2+v^2)$$

$$=a_0^2-\frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

⇒ Since a_0 is a known constant of the flow, this Equation gives a^2 as a function of ϕ

↳ This is because freestream conditions are given and we know that a_0 = const throughout the flow field.

· Therefore,

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↳ This is because freestream conditions are given and we know that a_0 = const throughout the flow field.

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0$$

; where $a_2 = a_0^2 - \frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]$

This equation represents a combination of the continuity, momentum, and energy equation.

- Since it is a single P.D.E. in terms of one dependent variable ϕ, in principle, it can be solved to obtain ϕ for the flow field.

2D, steady, isentropic, irrotational, and compressible

- However, we may have to solve the equation numerically.

- Once ϕ is known, all the other flow variables are directly obtained such as
$$\begin{cases}\phi \rightarrow u \text{ and } v\\ \phi \rightarrow a \rightarrow M\\ M \rightarrow T, \rho, \text{ and } p \text{ from Isentropic relation}\end{cases}$$

- Keep in mind that ;

within the assumptions, the full potential equation holds for all Mach numbers and for all 2D body shapes, thin and thick.

okay, let's talk about linearization.

· Consider 2D, irrotational, isentropic flow over an airfoil

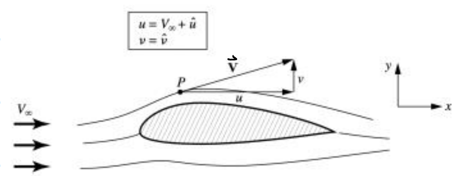

Figure 11.2 Uniform flow and perturbed flow.

· At an arbitrary point (P) in the flow field, the velocity $\vec{V}$ is with the x and y components given by u and v, respectively.

· Then, the sum velocity in x direction = uniform flow velocity (U_∞) + some extra increments in velocity ($\hat{u}$)

↳ These increments are called perturbations and it can be positive or negative.

· We can define a perturbation velocity potential ($\hat{\phi}$) such that ;

$$\begin{cases} u = U_\infty + \hat{u} \iff \frac{\partial\phi}{\partial x} = U_\infty + \frac{\partial\hat{\phi}}{\partial x} \iff \phi = U_\infty x + \hat{\phi}\\ v = \hat{v} \iff \frac{\partial\phi}{\partial y} = \frac{\partial\hat{\phi}}{\partial y}\end{cases}$$

$$\iff \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\hat{\phi}}{\partial y^2}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\hat{\phi}}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\hat{\phi}}{\partial x}\right)$$

· Substituting the above definition into full potential equation, we have

$$\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-\frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial x\partial y}=0$$

$$\iff \left[1-\frac{1}{a^2}\left(U_\infty+\frac{\partial\hat{\phi}}{\partial x}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial y^2}-\frac{2}{a^2}\left(U_\infty+\frac{\partial\hat{\phi}}{\partial x}\right)\left(\frac{\partial\hat{\phi}}{\partial y}\right)\frac{\partial^2\hat{\phi}}{\partial x\partial y}=0$$

$$\Leftrightarrow \left[1-\frac{1}{a^2}\left(U_\infty+\frac{\partial\hat{\phi}}{\partial x}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial x^2}+\left[1-\frac{1}{a^2}\left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial y^2}-\frac{2}{a^2}\left(U_\infty+\frac{\partial\hat{\phi}}{\partial x}\right)\left(\frac{\partial\hat{\phi}}{\partial y}\right)\frac{\partial^2\hat{\phi}}{\partial x\partial y}=0$$

; now, it's expressed in terms of $\hat{\phi}$ (instead of ϕ)

Here,

$$a_2 = a_0^2 - \frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

$$\Leftrightarrow a_2 = a_0^2 - \frac{\gamma-1}{2}\left[\left(U_\infty+\frac{\partial\hat{\phi}}{\partial x}\right)^2+\left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\right]$$

· Substitute a_2 into the perturbation full potential equation and algebraically rearranging, we have

$$(1-M_\infty^2)\frac{\partial\hat{u}}{\partial x}+\frac{\partial\hat{v}}{\partial y}=M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial\hat{u}}{\partial x}$$
$$+M_\infty^2\left[(\gamma-1)\frac{\hat{u}}{V_\infty}+\frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2}+\frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}\right]\frac{\partial\hat{v}}{\partial y}$$
$$+M_\infty^2\left[\frac{\hat{v}}{V_\infty}\left(1+\frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial\hat{u}}{\partial y}+\frac{\partial\hat{v}}{\partial x}\right)\right] \qquad (11.16)$$

· LHS = RHS

↓ Linear ↓ Non-linear

· Keep in mind that ;

the size of $\hat{u}$ and $\hat{v}$ can be large or small

· Now, let us limit our considerations to small perturbations

(That is, assume that the airfoil is a slender body at small angle of attack)

⇒ In this case, $\frac{\hat{u}}{U_\infty} \lll 1$, $\frac{\hat{v}}{U_\infty} \lll 1$, $\frac{\hat{u}^2}{U_\infty^2} \lll 1$, and $\frac{\hat{v}^2}{U_\infty^2} \lll 1$

· Finally, we have

$$(1-M_\infty^2)\frac{\partial^2\hat{\phi}}{\partial x^2}+\frac{\partial^2\hat{\phi}}{\partial y^2}=0$$

; Linear Partial Differential Equation

· However, it cannot be used for
- Transonic flow, where approximately, $0.8 < M_\infty < 1.2$
- Hypersonic flow, where $M_\infty > 5$

· This is because ;

- Small perturbation theory leads to a non-linear partial differential equation for $\hat{\phi}$.

Even if $\frac{\hat{u}}{U_\infty} \lll 1, \cdots$

· For example, so-called transonic flow where $M_\infty \simeq 1$

- For LHS, the coefficient of $\frac{\partial^2\hat{\phi}}{\partial x^2}$ becomes very small ⇒ For RHS, the term related to $\frac{\partial^2\hat{\phi}}{\partial x^2}$ cannot be neglected.

- For LHS, however, $M_\infty \simeq 1$ doesn't affect the term $\frac{\partial^2\hat{\phi}}{\partial y^2}$ ⇒ So, the term related to $\frac{\partial^2\hat{\phi}}{\partial y^2}$ in RHS can be neglected.

Large · x = Very ~~small~~ · x → 0

Small · x = Very small · x ?

· If the flow is hypersonic,

- For RHS,

$$LHS = \underbrace{M_\infty^{-}}_{\text{very large}} \left[\underbrace{\frac{\quad}{U_\infty}}_{\text{very small}} + \cdots \right] \frac{\quad}{dx^2}$$

$\therefore$ It is not possible to cancel out.

cf. Pressure coefficient

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) \rightarrow \text{Linearization} \rightarrow C_p = -\frac{2\hat{u}}{U_\infty}$$

① Derivation of linearized form for the pressure coefficient.

Def. $C_p \equiv \dfrac{p - p_\infty}{q_\infty}$; where $q_\infty = \frac{1}{2}\rho_\infty V_\infty^2$

Since ~~(Due to) ρ is variable for compressible flow~~, Rearrange ..

↳ 동압에 대한 표현을 M으로 바꾸면 여러모로 편하겠다!

$$q_\infty = \frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2}p_\infty\left(\frac{\rho_\infty}{\gamma p_\infty}\right)V_\infty^2$$

Using $a^2 = \dfrac{\gamma p}{\rho}$, $M = \dfrac{V}{a}$, we have ..

$$q_\infty = \frac{\gamma}{2}p_\infty\frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2}p_\infty M_\infty^2$$

Then, Substituting above Equation into Def. of C_p ..

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)}$$; pressure coeff. expressed in terms of M_∞

To obtain a linearized form, recall the Equation ..

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \Leftrightarrow c_p T_0 = \text{const}$$

Then, $T + \dfrac{V^2}{2c_p} = T_\infty + \dfrac{V_\infty^2}{2c_p}$; where $c_p = \dfrac{\gamma}{\gamma-1}R$

$$\Leftrightarrow T - T_\infty = \frac{V_\infty^2 - V^2}{2\gamma R/(\gamma-1)}$$

Using $a_\infty = \sqrt{\gamma R T_\infty}$, we have

$$\frac{T}{T_\infty} - 1 = \frac{\gamma-1}{2}\frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma-1}{2}\frac{V_\infty^2 - V^2}{a_\infty^2}$$; where $V^2 = (V_\infty + \hat{u})^2 + \hat{v}^2$

Since the flow is isentropic, we can use isentropic relation.

$$\frac{p}{p_\infty} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2)\right]^{\gamma/(\gamma-1)}$$

$$\frac{p}{p_\infty} = \left[1 - \underline{\frac{\gamma - 1}{2}M_\infty^2\left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right)}\right]^{\gamma/(\gamma-1)}$$

↳ from perturbation assumption. it is very small.

Let —— $= \epsilon$: ϵ is very small value.

Then, $\frac{p}{p_\infty} = (1 - \epsilon)^{\frac{\gamma}{\gamma - 1}}$

from the binomial expansion, neglecting Higher-order terms,

$$\left(\frac{p}{p_\infty}\right) = 1 - \frac{\gamma}{\gamma - 1}\epsilon + (\quad)\cancel{\epsilon^2} + (\quad)\cancel{\epsilon^3} + \cdots$$

$\to 0$ $\to 0$

※ 약) 이항확장 (binomial Expansion)

: $(P+Q)^N$ 의 확장, 확장에서 $N+1$항 각각은 확률을 가지며, 모든 확률의 합은 1이다.

Anyway..

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\left(\frac{p}{p_\infty}\right) - 1\right)$$

$$C_p = \frac{\cancel{2}}{\gamma M_\infty^2}\left[\cancel{1} - \frac{\cancel{\gamma}}{\cancel{2}}M_\infty^2\left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right) + \cancel{\cdots}^{\,0} - \cancel{1}\right]$$

$$C_p = -\frac{2\hat{u}}{V_\infty} + \cancel{\frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}}$$

↳ 0 (∵ Small perturbation) 작은걸 제곱했으니, 더 작지~..

Finally..

$$\boxed{C_p \cong -\frac{2\hat{u}}{V_\infty}}$$

"Linearized pressure coefficient"

: x 방향의 perturbation velocity 에 2배를 해서 free stream velocity (V_∞) 로 나눈게 Linearized pressure coefficient 다.

(It is valid only for small pertubations.)